

could prove its own consistency.” Since \prec is arithmetically definable, Gentzen’s induction principle can be expressed as a “schema” in the language of \mathbf{PA} in the same manner as ordinary induction. In the system obtained by adding this principle to \mathbf{PA} (call it $\mathbf{PA}+$), Gentzen’s consistency proof for \mathbf{PA} can certainly be carried out. But this is not an instance of a “theory which could prove its own consistency;” the consistency of \mathbf{PA} is proved in a different system $\mathbf{PA}+$. There is also a misstatement on p. 215: the authors surely meant to say that it *was* clear to Gödel that the primitive recursive functions were *not* “all the computable ones”

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Linear operators in spaces with an indefinite metric, by T. Ya. Azizov and I. S. Iokhvidov. John Wiley & Sons, 1989, 300 pp., \$82.95. ISBN 0-471-92129-7

Let H be a Hilbert space (over the complex numbers), and let J be a bounded linear selfadjoint operator on H such that $J^2 = I$. Consider the sesquilinear form $[\cdot, \cdot]$ induced by J :

$$[x, y] = \langle Jx, y \rangle, \quad x, y \in H,$$

where $\langle \cdot, \cdot \rangle$ stands for the scalar product in H . The corresponding quadratic form $[x, x]$ is indefinite (unless $J = I$ or $J = -I$), in other words, there exist $x, y \in H$ for which $[x, x] < 0$ and $[y, y] > 0$. The space H , together with the sesquilinear form $[\cdot, \cdot]$ generated by some J as above, is commonly called a *Krein space*. One can also define the Krein spaces intrinsically, by starting with a topological vector space and a continuous sesquilinear form on it, and by imposing suitable completeness and nondegeneracy axioms. The reviewed book is devoted to the geometry of Krein spaces and the spectral structure and related properties of several important classes of bounded and unbounded linear operators on Krein spaces.

1. THE SUBJECT

Why Krein spaces? As with many mathematical disciplines, there are two compelling reasons: (1) important applications in

mathematics and the physical sciences; (2) intrinsic depth and beauty of the subject. Historically, finite-dimensional Krein spaces (in disguise) appeared in the nineteenth century, with the development of noneuclidean geometries. I mention just one such geometry: Minkowski space of the special relativity theory, which is a 4-dimensional Krein space (over the real numbers), with $J = \text{diagonal}(1, 1, 1, -1)$. Another 19th century problem (solved by Weierstrass [We] and Kronecker [K]) where finite-dimensional Krein spaces appear naturally, is the classification of pairs of hermitian $n \times n$ matrices (A, B) up to simultaneous congruence (two pairs of hermitian matrices (A, B) and (A', B') are called simultaneously congruent if $A' = S^*AS$, $B' = S^*BS$ for some nonsingular matrix S). Assuming that B is nonsingular, by applying simultaneous congruence to A and B , we can further assume that $B^2 = I$. If $[\cdot, \cdot]$ is the sesquilinear form induced by B in \mathbb{C}^n , then the matrix $X := B^{-1}A$ has the property that $[Xx, y] = [x, Xy]$, $x, y \in \mathbb{C}^n$, i.e. X is *selfadjoint* with respect to $[\cdot, \cdot]$ (in short, *B-selfadjoint*). Recently, there is renewed interest in finite-dimensional Krein spaces, motivated in large part by engineering applications (see [GLR, BGR]).

It was only in the 1940s, however, when the first works on linear operators in an infinite dimensional indefinite Krein space appeared, starting with the ground-breaking paper by Pontryagin [P]. Significantly, the original motivation for this paper came from a problem in mechanics (motion of a fluid in a container) posed by S. L. Sobolev.

Since then the theory of linear operators in Krein spaces has been developed into a major branch of modern operator theory. M. G. Krein had profoundly influenced and inspired this development, and the Krein spaces are justly named after him. Krein space is a natural place for possible generalizations and new applications of many well-known results and techniques in operator theory. Several research monographs, lecture notes, survey articles, and hundreds of papers appeared, devoted entirely to this subject. I will describe below some of the more significant directions of research here. In what follows, H is the Krein space with the sesquilinear form $[\cdot, \cdot]$ induced by a bounded linear operator J ($J^* = J$, $J^2 = I$, and to avoid the more familiar Hilbert space situation it will be assumed $J \neq \pm I$).

Invariant maximal positive semidefinite subspaces. The quest for (nontrivial) invariant subspaces of operators is one of the ma-

major thrusts of research in modern operator theory (see the review [B] for description of some of the major recent advances in this area). In the Krein space framework, it is of interest to consider invariant subspaces with additional properties (such as the positive semidefiniteness defined below) with respect to $[\cdot, \cdot]$. This interest stems from applications; for example, invariant maximal positive semidefinite subspaces give rise to solutions of the operator equations

$$Z^2 + BZ + C = 0$$

with selfadjoint coefficients B and C , which appears in the theory of small damped vibrations of continua [KL].

A (closed) subspace $L \subset H$ is called *positive semidefinite* if $[x, x] \geq 0$ for every $x \in L$. A *maximal positive semidefinite* subspace is, by definition, a positive semidefinite one which is also a maximal (with respect to the set theoretical inclusion) subspace having this property.

In the following theorem P^+ stands for the projector on the kernel of $J - I$ along the kernel of $J + I$ (note that the conditions $J = J^*$, $J^2 = I$, $J \neq \pm I$ imply that $\text{Ker}(J - I)$ and $\text{Ker}(J + I)$ are orthogonal complements to each other).

Theorem 1. *Let $V: H \rightarrow H$ be a linear densely defined operator such that $[Vx, Vx] \geq [x, x]$, $x \in H$ and the operator*

$$P^+V(I - P^+)$$

is compact.

Then there exists a subspace $L \subset H$ which is simultaneously V -invariant and maximal positive semidefinite.

The first version of Theorem 1(ii) was proved by Krein [K1], other results of this type are found in [I, W], and, in a very general form, in the reviewed book.

In particular, if either P^+ or $I - P^+$ is a finite rank operator (Krein spaces with this property are called *Pontryagin spaces*) then the operator $P^+V(I - P^+)$ is obviously compact, and therefore any V satisfying the property (ii) has an invariant maximal positive semidefinite subspace. This fact (for bounded selfadjoint operators with respect to $[\cdot, \cdot]$) was observed already in [P]. In connection with Theorem 1 I would like to mention an old and well-known unsolved problem: Does every bounded selfadjoint operator in Krein space have an invariant maximal semidefinite

subspace? As far as I know, even the existence of a nontrivial invariant subspace for every such operator is an open question.

Spectral functions for J -selfadjoint operators. The spectral theorem for selfadjoint (possibly unbounded) operators in Hilbert space is a well-known and classical result, part of the general theory of unbounded operators in Hilbert space developed originally by von Neumann [vN] and Stone [S]. The first motivations and applications of this theory, especially extensions of symmetric operators, came from differential operators and boundary-value problems. (A linear operator A with dense domain of definition $D(A)$ in the Hilbert space H is called *symmetric* if

$$\langle Ax, y \rangle = \langle x, Ay \rangle$$

for all $x, y \in D(A)$; an *extension* of A is a symmetric operator B with $D(B) \supset D(A)$; finally, a symmetric operator A is called *selfadjoint* if $D(A) = D(A^*)$, where A^* is the operator defined by

$$\langle Ax, y \rangle = \langle x, A^*y \rangle$$

for all $x \in D(A)$, with the natural domain of definition $D(A^*)$.) Up to now there is no complete analogue of this theory for general J -selfadjoint operators (even the bounded ones). One major difficulty is that the spectrum of a J -selfadjoint operator need not be real in contrast with selfadjoint operators in a Hilbert space. However, for a special class of J -selfadjoint operators— J -definitizable operators—a spectral function (described in Theorem 2) is available. To somewhat simplify the exposition we assume here that the operators are bounded, although the spectral function exists for unbounded J -definitizable operators as well.

A (bounded) J -selfadjoint operator A is called *J -definitizable* if $[p(A)x, x] \geq 0$ for all $x \in H$, where $p(\lambda)$ is a fixed polynomial with real coefficients (called *definitizing polynomial* for A). A finite set $c(A)$ is introduced for every J -definitizable operator A consisting of common real roots of all definitizing polynomials of A . Let R_A be the set of bounded intervals with endpoints not in $c(A)$ and complements of such intervals. $\bar{\Delta}$ stands for the closure of Δ .

Theorem 2. *Let A be a (bounded) J -definitizable operator on the Krein space H . Then there exists a mapping (spectral function) $E: R_A \rightarrow \{\text{bounded } J\text{-selfadjoint operators on } H\}$ with the following properties:*

$$(i) \quad E(\Delta)E(\Delta') = E(\Delta \cap \Delta'), \quad \Delta, \Delta' \in R_A,$$

- (ii) $E(\Delta \cup \Delta') = E(\Delta) + E(\Delta')$ if $\Delta, \Delta' \in R_A, \Delta \cup \Delta' \in R_A, \Delta \cap \Delta' = \emptyset,$
- (iii) if $p > 0$ (respectively, $p < 0$) on $\bar{\Delta}$ for some definitizing polynomial $p(\lambda)$ of A and some $\Delta \in R_A,$ then $[x, x] > 0$ (respectively, $[x, x] < 0$) for all nonzero $x \in E(\Delta)H,$
- (iv) $E(\Delta)$ commutes with every (bounded) linear operator T that commutes with $A.$

The result of Theorem 2 for unbounded operators appeared first in [L1]; (see also [L2, J]).

As a corollary, the formula (which is perhaps more familiar) is obtained:

$$AE(\Delta) = \int_{\Delta} t dE(t)$$

for every interval $\Delta \in R_A$ that does not intersect $c(A).$

It is worth mentioning that the theory of extensions of J -isometric and J -symmetric operators is well developed by now, and a substantial part of the reviewed book is devoted to this theory.

2. APPLICATIONS

The applications of linear operators in Krein spaces are numerous, and it would be impossible to mention here all of them, or even most of them. I will focus on a few topics (differential and integral equations, interpolation, systems theory), and leave aside many other equally important applications, such as orthogonal polynomials, moment problems, reproducing kernel spaces, noneuclidean geometry, etc.

Differential and integral equations. One of the first applications of the theory of extensions of J -symmetric operators to boundary value problems for hyperbolic systems of partial differential equation is given in [Ph]. I mention only a few illustrative examples of applications of the Krein space theory to differential and integral equations.

A variety of steady-state transport phenomena (including neutrons, electrons, rarefied gases, etc.) can be represented by equation of the form

$$(2.1) \quad (T\psi)'(x) = -(A\psi)(x), \quad 0 < x < \infty$$

with boundary conditions

$$(2.2) \quad (Q_+ \psi(0) = f_+, \|\psi(x)\| \text{ bounded as } x \rightarrow \infty.$$

Here T and A are selfadjoint operators on a Hilbert space H , and Q_+ is a suitable projector. It is assumed that $\text{Ker } T = \{0\}$ and A is invertible. Typically in applications, $H = L^2[-1, 1]$, T is the multiplication operator $(Tf)(t) = tf(t)$, and $I - A$ is an integral operator. The case when $A > 0$ corresponds to the subcritical (nonmultiplying) medium of transport. The interesting case here is when A has a finite number of negative eigenvalues which corresponds to the supercritical, or multiplying, medium (this case is especially important in neutron transport). A fruitful way to approach the transport equation (2.1) is by introducing an indefinite scalar product $[x, y] = \langle Ax, y \rangle$ and by realizing that $A^{-1}T$ is A -selfadjoint. Therefore, its inverse $T^{-1}A$ can be studied using the theory of selfadjoint operators in Krein spaces (in this example, even Pontryagin spaces). This approach has been used, for example, in [BG, GMP].

A Krein space structure appears naturally in the study of Sturm-Liouville operators

$$-(p(x)^{-1}y'(x))' + q(x)y(x) = \lambda r(x)y(x), \quad x \in [0, b]$$

with suitable boundary conditions, where the weight function $r(x)$ changes sign. Here the indefinite scalar product is given by

$$[f, g] = \int_0^b f(x)\overline{g(x)}r(x) dx,$$

where f, g belong to the Hilbert space of square integrable functions with weight $|r(x)|$. It turns out that under suitable condition, the Sturm-Liouville operator is J -selfadjoint and J -definitizable, so the theory of J -definitizable operators can be successfully applied to obtain spectral decompositions, expansion theorems [DL].

Systems theory and control. Consider a 2-port wave scattering system with incoming wave y_1 and outgoing wave x_1 in the first port, and incoming wave x_2 and outgoing wave y_2 in the second port (all waves are represented by column vector valued functions of the complex variable z , where $\text{Re } z \geq 0$). The input-output relation of this system (under certain simplifying assumptions) is described by

$$\begin{bmatrix} x_1 \\ y_2 \end{bmatrix} = \Sigma \begin{bmatrix} y_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ x_2 \end{bmatrix}$$

and the port-to-port relation is described by

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \Theta \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}.$$

Here Σ and Θ are matrix valued functions of z with rational entries of requisite sizes, known as the *scattering matrix* and the *chain-scattering matrix* of the system, respectively. The stability considerations (bounded inputs result in bounded outputs) can be expressed by assuming that all poles of Σ are in the open left half-plane. If, furthermore, Σ_{21} is invertible, then the scattering matrix and the chain-scattering matrix are related by the *Redheffer transformation* [Re]

$$\Theta = \begin{bmatrix} \Sigma_{12} - \Sigma_{11}\Sigma_{21}^{-1}\Sigma_{22} & \Sigma_{11}\Sigma_{21}^{-1} \\ -\Sigma_{21}^{-1}\Sigma_{22} & \Sigma_{21}^{-1} \end{bmatrix}.$$

The energy conservation law leads to the equality

$$x_1(z)^* x_1(z) + y_2(z)^* y_2(z) = y_1(z)^* y_1(z) + x_2(z)^* x_2(z)$$

for z on the imaginary axis, which translates into

$$(2.3) \quad (\Theta(z))^* J \Theta(z) \leq J, \quad \operatorname{Re} z \geq 0,$$

$$(2.4) \quad (\Theta(z))^* J \Theta(z) = J, \quad \operatorname{Re} z = 0,$$

where $J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$. (We use the notation $X \leq Y$ to indicate that the difference $Y - X$ is positive semidefinite). In other words, $\Theta(z)$ takes *J-unitary* values on the imaginary axis and is *J-contractive* in the open right-hand size Π^+ . So the Krein space structures come into the problem.

In many engineering problems a controller is sought in order to satisfy design specifications. Mathematically, the controller is represented by the outgoing-to-incoming wave transformation in port 2, i.e. by putting $x_2 = Ky_2$. This leaves the system with just one incoming wave y_1 and one outgoing wave x_1 , where the transformation from y_1 to x_1 is given in terms of the linear fractional map

$$x_1 = W y_1,$$

where

$$(2.5) \quad W = (\Theta_{11}K + \Theta_{12})(\Theta_{21}K + \Theta_{22})^{-1}.$$

A typical design problem in this context is the "standard problem" in the H_∞ -control: Find a controller K such that W is stable (all the poles in the open left half-plane) and the H_∞ -norm

$$\|W\|_\infty = \sup_{s \in \mathbf{R}} \|W(is)\|$$

is minimal. H_∞ here refers to the Hardy space of bounded analytic matrix functions on the open right half-plane. One interpretation of $\|W(is)\|$ is the sensitivity of the controlled system to disturbances having frequency s . Thus, the H_∞ -control can be understood as a min-max optimization problem, or the worst case optimization. It turns out that the suboptimal solutions of the standard problem are described essentially by (2.5) where Θ is a rational matrix function taking J -unitary values on the imaginary axis and judiciously chosen to satisfy certain interpolation conditions, while K is any rational matrix function with all poles in the open left half-plane and $\|\cdot\|_\infty$ norm less than 1. I should point out that the interpolation approach to H_∞ -optimization shown here is only one of the many ways this problem is attacked and solved (the reader may consult [F,D,H,BGR]). The H_∞ -optimization problem, in its many versions and generalizations, is one of the most important in modern electrical engineering, both from practical and theoretical points of view.

Interpolation. Recently, there is a surge of renewed interest in extensions and generalizations of classical function theoretic interpolation problems. A major driving force behind this revival are applications in systems and control, such as the H_∞ -control problem described above.

As a typical example, consider a version of the classical Nevanlinna-Pick problem: Given a sequence (finite or infinite) of points $\{a_i\}$ in the open right half-plane Π^+ and a corresponding sequence of complex numbers $\{w_i\}$ find, if possible, all functions $f(z)$ which are analytic in the open right half-plane, bounded in norm by 1, and such that $f(z_i) = w_i$ for all i . There are many conceivable ways this problem can be generalized to matrix valued functions. One relatively simple generalization is the following: Given finite number of points a_1, \dots, z_n in the open right half-plane Π^+ , $1 \times M$ row vectors x_1, \dots, x_n and $1 \times N$ row vectors y_1, \dots, y_n find an $M \times N$ matrix function $F(z)$ which is analytic in Π^+ , bounded in norm there by a constant less than 1, and satisfies

$$x_i F(z_i) = y_i, \quad i = 1, \dots, n.$$

It turns out that the solution $F(z)$ exists if and only if the $n \times n$ matrix $[(z_i + \bar{z}_j)^{-1} (x_i x_j^* - y_i y_j^*)]_{i,j=1}^n$ is positive definite, and in such case all solutions $F(z)$ can be described by the linear-

fractional map

$$F(z) = (\Theta_{11}(z)G(z) + \Theta_{12}(z)) (\Theta_{21}(z)G(z) + \Theta_{22}(z))^{-1},$$

where $G(z)$ is analytic in Π^+ with $\sup_{z \in \Pi^+} \|G(z)\| < 1$ and

$$\Theta(z) = \begin{bmatrix} \Theta_{11}(z) & \Theta_{12}(z) \\ \Theta_{21}(z) & \Theta_{22}(z) \end{bmatrix}$$

is a fixed $(M+N) \times (M+N)$ matrix function which satisfies (2.3), (2.4) and certain interpolation conditions. Thus, the Krein space shows up again.

There are many ways in which one could study the Nevanlinna-Pick and related problems and their generalizations. The Krein space techniques play a crucial role in the Ball-Helton theory which is based on representations of invariant subspaces for the shift operator (see [BH, H] and references there), and in the approach based on the reproducing kernel Hilbert spaces (see [D]).

3. THE BOOK

The reviewed book occupies a unique place in the literature and fills in a substantial gap. This is the only existing monograph offering a comprehensive study of Krein space geometry and linear operators (bounded and unbounded) in Krein spaces. It should be noted that this area of operator theory has been developed mostly in the Soviet Union and East Europe, and the bibliography given in the book fully reflects this fact (about 203 references out of a total of 231 are the works of Soviet and East European mathematicians). By comparison, the well-known book by Bogner [Bo] treats for the most part geometry of vector spaces with an indefinite scalar product which are more general than Krein spaces; on the other hand, [IKL] is devoted entirely to Pontryagin spaces, [GLR] concerns finite dimensional Krein spaces only, the lecture notes [A] are geared mainly towards results on invariant semidefinite subspaces, while the main concern in [L2] is the spectral function. Some of the earlier lectures and survey articles [IK, K2], while still of research interest, are of limited scope and in certain aspects out-of-date. Very appropriately, more than half of the reviewed book is taken up by geometry of Krein spaces and basic properties of fundamental classes of linear operators; the other part of the book is devoted to more advanced and specialized topics: invariant semidefinite subspaces, spectral topics (including completeness of root vectors), extensions of J -isometric and J -symmetric operators.

Azizov and Iokhvidov did superb scholarly work. The authors are leading experts in the field, and it shows. (Sadly, only one of the authors is alive today: Iokhvidov passed away in 1984.) Within about 300 pages, major research directions in the theory of linear operators in Krein space are presented starting from the basics, and many results are given in full generality, with complete proofs. Exercises contain a wealth of additional information. Of course, this does not come about without a price. Thus, the style is often laconic, especially in the second part of the book, and, without apologies, the reader's knowledge in many advanced topics in operator theory is taken for granted. Nevertheless, the book can serve as an excellent graduate topics course if guided by an expert in the subject.

To put the contents of the book in perspective, I should mention that in-depth discussions of important applications (some material on applications is touched upon in §4.3) are left out of the book by purpose. The authors justifiably point out that such discussions would dramatically increase its size. Although numerous applications of geometry and linear operators in Krein spaces can be found in the monographs [DK, GLR, R, M, D, H] and extensive articles [IK, AI], it would be highly desirable to have the applications collected and explained from a unified point of view (based on the reviewed book). Perhaps a monograph written along these lines will appear some day.

Unfortunately, the translation and proofreading leave much to be desired. The book did not escape many common traps when translating from Russian. Thus, we find "G. Langer" for "H. Langer," "carrier" for "support," "ring" for "annulus," "paragraph" for "section," etc. Misspelled names are abundant; "Cagley-Neyman" for "Cayley-von Neumann" (p. 163) is perhaps the most striking occurrence. More importantly, several theorems are misstated (in Theorem 1.8.15, " $1 \notin$ " is left out in part (b), " $1 \in$ " is left out in part (c); in Theorem 4.3.7, " N_λ " should be replaced by " M_λ " in part 4); in Theorem 2.5.23 "stable" should be replaced by "is strongly stable"; in Lemma 4.2.5 it should be " $A \in K(H)$ having the set of properties K "). Some terminology is outdated, unusual or somewhat illogical: "bicomact" should be "compact," "dilatation" should be "dilation"; "non-contractive operator" does not mean "operator which is not a contractive one". All these misspellings, misstatements, etc. are easily spotted and corrected by an expert; however, some of them might be serious stumbling blocks for a student in the subject.

Despite these shortcomings, the book will surely serve as the standard reference guide in a vital area of operator theory for years to come. It represents a summary of a long mathematical tradition which for some time was relatively little known in the West. In recent years, however, the Krein space theory and its applications are considerably more appreciated among Western mathematicians, and this fact makes the reviewed book especially timely and valuable. It belongs on the bookshelf of every operator theorist, and, given the vast array of applications, of many other mathematicians, and some physicists and electrical engineers as well.

REFERENCES

- [A] T. Ando, *Linear operators on Krein spaces*, Sapporo, Japan, 1979.
- [AI] T. Ya Azzizov and I. Iokhvidov, *Linear operators in spaces with an indefinite metric*, Mathematical Analysis 17 (Itogi nauki i tekhniki), 113–205, Moscow, 1979; (English Transl. J. Soviet Math. 15 (1981), 438–490. (Russian))
- [B] H. Bercovici, *Review on Introduction to operator theory and invariant subspaces* by Bernard Beauzamy, Bull. of Amer. Math. Soc. (N.S.) 22 (1990), 148–152.
- [BG] J. A. Ball and W. Greenberg, *A Pontryagin space analysis of the supercritical transport equation*, Transport Theory and Stat. Physics 4 (1975), 143–154.
- [BGR] J. A. Ball, I. Gohberg, and L. Rodman, *Interpolation of rational matrix functions*, OT45, Birkhäuser, 1990.
- [BH] J. A. Ball and J. W. Helton, *A Beurling-Lax theorem for the Lie group $U(m, n)$ which contains most classical interpolation theory*, J. Operator Theory 9 (1983), 107–142.
- [Bo] J. Bognar, *Indefinite inner product spaces*, Springer-Verlag, Berlin, 1974.
- [D] H. Dym, *J-contractive matrix functions, reproducing kernel Hilbert spaces and interpolation*, CBMS Regional Conf. Ser. in Math., vol. 71, Amer. Math. Soc., Providence, RI, 1989.
- [DK] Yu. L. Daletskiy and M. G. Krein, *Stable solutions of differential equations in a Banach space* (Russian), Nauka, Moscow, 1970; (English Transl.: Amer. Math. Soc. Transl. Math. Monographs, vol. 43, Providence, RI, 1974.
- [DL] K. Daho and H. Langer, *Sturm-Liouville operators with an indefinite weight function*, Proc. Royal Soc. Edinburgh 78A (1977), 161–191.
- [Fr] B. Francis, *A course in H_∞ control theory*, Lecture Notes in Control and Information Sciences, vol. 88, Springer-Verlag, Berlin and New York, 1987.
- [GLR] I. Gohberg, P. Lancaster, and L. Rodman, *Matrices and indefinite scalar products*, OT8, Birkhäuser, Verlag, Basel, 1983.
- [GMP] W. Greenberg, C. V. M. van der Mee, and V. Protopopescu, *Boundary value problems in abstract kinetic theory*, OT23, Birkhäuser, Basel, 1987.

- [H] J. W. Helton, *Operator theory, analytic functions, matrices and electrical engineering*, CBMS Regional Conf. Ser. in Math., vol. 68, Amer. Math. Soc., Providence, RI, 1987.
- [I] I. S. Iokhvidov, *On a lemma of K. Fan generalizing A. N. Tikhonov's principle of fixed points*, Dokl. Akad. Nauk USSR **159** (1964), 501–504. (Russian)
- [IK] I. S. Iokhvidov and M. G. Krein, *Spectral theory of operators in spaces with an indefinite metric*. I (Russian), Trudy Moskov., Mat. Obsch. **5** (1965), 367–432; Amer. Math. Soc. Translations (2), 1960, 105–75; II (Russian), Trudy Moskov, Mat. Obschch, **8** (1969), 413–96; Amer. Math. Soc. Translations (2), **34**, 283–373.
- [IKL] I. S. Iokhvidov, M. G. Krein, and H. Langer, *Introduction to the spectral theory of operators in spaces with an indefinite metric*, Akademie-Verlag, Berlin, 1982.
- [J] P. Jonas, *On the functional calculus and the spectral function for definitizable operators in Krein space*, Beiträge Anal. **16** (1981), 121–135.
- [K1] M. G. Krein, *A new application of the fixed-point principle in the theory of linear operators in a space with an indefinite metric*, Dokl. Akad. Nauk USSR 1964 **154** (5) 1023–1026; (English Transl.: Soviet Math. Dokl. **5** (1964), 224–227. (Russian)
- [K2] ———, *Introduction to the geometry of indefinite J -spaces and the theory of operators in those spaces*, Second Summer Mathematical School, I, Kiev, 1965, 15–92; English Transl: AMS Transl. (2) **93** (1970), 103–176. (Russian)
- [K] L. Kronecker, *Ueber die congruenten Transformationen der bilinearen Formen*, Monatsber. Akad. Wiss., Berlin, 1874, 397–447.
- [KL] M. G. Krein, and H. Langer, *On some mathematical principles in the linear theory of damped oscillations of continua*, Trudy mezhdunarod. simposiuma po prilozheniyam teorii funktsiy v mehanike sposhnoy sredy, vol. 2, Moscow, 1966, 283–322; (English transl.: Integral Equations and Operator Theory **1** (1978), 364–399; 539–566.
- [L1] H. Langer, *Spectral theory of linear operators in J -spaces and some applications to the bundle $L(\lambda) = \lambda^2 I + \lambda B + C$* , Habilitationsschrift, Dresden, 1965. (German)
- [L2] ———, *Spectral functions of definitizable operators in Krein spaces*, Lecture Notes in Math., vol. 948, 1982, pp. 1–46.
- [M] A. S. Markus, *Introduction to spectral theory of polynomial operator pencils*, Stiinca, Kishinev, 1986; (English Transl.: Amer. Math. Soc. Transl. of Math. Monographs, vol. 71, 1988. (Russian)
- [P] L. S. Pontryagin, *Hermitian operators in spaces with indefinite metric*, Izvestiya Akad. Nauk USSR, Ser. Matem. **8** (1944), 243–280. (Russian)
- [Ph] R. Phillips, *Dissipative operators and hyperbolic systems of partial differential equations*, Trans. Amer. Math. Soc. **90** (1959), 193–254.
- [R] L. Rodman, *An introduction to operator polynomials*, OT38, Birkhäuser, Basel, 1989.
- [Re] R. M. Redheffer, *On a certain linear fractional transformation*, J. Math. Phys. **39** (1960), 260–286.
- [S] M. H. Stone, *Linear transformations in Hilbert space and their applications to analysis*, Amer. Math. Soc. Colloq. Publ., vol. 15, New York, 1932.

- [vN] J. von Neumann, *Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren*, Math. Ann. **102** (1929-1930), 49–131.
- [W] G. Wittstock, *On invariant subspaces of positive transformations in spaces with indefinite metric*, Math. Ann. **172** (1967), 167–175. (German)
- [We] K. Weierstrass, *Zür Theorie der bilinearen und quadratischen Formen*, Monatsber. Akad. Wiss., Berlin, 1868, pp. 310–338.

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1. INTRODUCTION

Central to Banach Space theory is the study of the classical Banach spaces c_0 , ℓ^p , L^p ($1 \leq p \leq \infty$), $C(K)$, and of their relationship with general Banach spaces.

The space ℓ^1 is of special importance in the theory of general Banach spaces. This is due to a phenomenon of considerable interest, namely that many pathological properties of Banach spaces are closely related to the fact that they have subspaces close to ℓ^1 . This is true in the “local” theory of Banach spaces (see e.g. Pisier’s work [P] for a celebrated example) as well as in infinite dimensional theory, the subject of our present interest.

Among the “elementary” spaces c_0 , ℓ^p , $p < \infty$, ℓ^1 is the only one that has a nonseparable dual. If a Banach space contains ℓ^1 , its dual is nonseparable. (For simplicity, we say that a Banach space contains ℓ^1 if it contains a subspace isomorphic to ℓ^1 .) It was conjectured for a long time that the converse holds. This conjecture was disproved in 1974 by a deep example of R. C. James, the so called James tree space JT. (This space and its variations remain of considerable interest.) Thus it came as a surprise that simple criteria allow one to decide whether or not a Banach space contains a copy of ℓ^1 . The main results in that direction were proven in 1974 by H. P. Rosenthal, and surely constitute one of the most beautiful achievements of Banach space theory. One of