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Probabilités et potentiel (Chapters XII–XVI), by Claude Dellacherie and Paul-André Meyer. Hermann, Paris, 1987, 374 pp. ISBN 2-7056-1417-6

In its current incarnation, *Probabilités et Potentiel* is a five-volume book, the last volume of which has not yet appeared. This fourth volume, subtitled "Théorie du potentiel associée à une résolvante. Théorie des processes de Markov," brings *Probabilités et Potentiel* to an impressive 1,372 pages. Although appearing in "installments," the text is very cohesive, laced with references to previous sections or chapters (and sometimes to future ones), and complemented with helpful remarks, "commentaries," and examples. The authors are in complete control of their subject, and the result is masterful. Meyer and Dellacherie have built their careers at the interface between probability and potential theory, and are responsible not only for creating a good deal of new mathematics there, but also for energetically propagating the ideas and techniques among members of the probability community via the proceedings of the (Strasbourg) Séminaire de Probabilités, and through their books. This current project is a *tour de force*. We'll start with a quick look at the project as a whole.

THE FAMILY HISTORY

This family of volumes has a history. Its *raison-d'être* is the existence of an intimate connection between probability theory and potential theory, whose discovery inspired a burst of mathematical activity during the 1950s and early 60s, with probability supplying new methods for potential theory and potential theory suggesting new directions for probability. Meyer set out to write a research monograph on potential theory, with the goal of exposing the new probabilistic techniques (especially martingale theory) to a wider audience of analysts. The book would also collect for probabilists the analytic tools (such as the theories of analytic sets and capacities) needed to exploit the potential theoretic approach. The resulting first edition of *Probabilités et Potentiel* [7] was a "slim" 266 pages consisting of three parts: A. Introduction to Probability Theory, B. Martingale Theory, and C. Analytic Potential Theory. (A second volume to be devoted to Markov processes was planned, which would tie together the various topics of the first volume by illustrating their applications to the theory of Markov processes. This second volume appeared a year later in draft form [8], but the final form is only now materializing in the second edition; the original project was sidelined by the publication of [1] in 1968.)

Probabilités et Potentiel was an influential book and, together with Dellacherie's *Capacitiés et Processus Stochastiques* [2], became one of the cornerstones of the "general theory of (stochastic) processes." This theory led to remarkable advances in Markov process theory, revolutionized stochastic differential equations and martingale theory, and provided a framework for probabilistic potential theory. In turn, the general theory of processes has grown to meet the demands of these (and other) areas. Dellacherie and Meyer say, in the preface to the first volume of the new edition of *Probabilités et Potentiel*, "the rapid evolution of the whole theory has discouraged us from building on the old foundations, and the support of an active mathematical environment has been an incentive to undertake again the full work from the start." The first volume of the second edition (Chapters I through IV) was published in 1975 (in English in 1978). In the new edition, each of the three parts of the first edition has become an entire volume (at least in the English translation—in the French, two chapters of Part C spill over into the fourth volume). The treatment of Markov processes (Part D) begins midway through the fourth (French) volume, and will be continued in the fifth.

PROBABILITY

A random variable X is a measurable mapping from a probability space into a state space E. The image of the probability measure under X is a measure on E called the distribution, or *law*, of X. Many properties of X are determined by its law, which is a basic tool of statisticians. A family of random variables (X_t) indexed by a parameter $t \in T$ is called a stochastic process. (T is usually a subset of the nonnegative integers or real numbers, and is thought of in these cases as "time.") Stochastic processes can also be studied "in law," although the situation is more complicated than for a single random variable because now we are working with a (possibly infinite, even uncountable) family of variables. The law of the *process* is actually a measure on the space of E-valued sequences or functions, and incorporates joint distributions for the family of variables. For certain classes of stochastic processes, analytic tools have been developed to aid in this study. For instance, real-valued Gaussian processes are studied largely in terms of the autocovariance function $v(s, t) = E(X_s X_t)$, which is a positive-definite function on $T \times T$. Another kind of function is used in the study of Markov processes.

Markov processes are characterized by the property that the "past and future are conditionally independent given the present." This means, even when the entire "past" $\{X_u, 0 \leq u \leq t\}$ is known, the best prediction of the "future" X_{s+t} uses only knowledge of X_t (the "present"). Because of this property, we can express the joint distributions of the process in terms of the *transition function* $P_{s,t}(x, dy)$, which gives the distribution of X_t, conditional on the knowledge that $X_s = x \in E$ $(s \leq t)$. (For homogeneous processes, the usual case, this distribution depends only on $t - s$.) Use of transition functions to study Markov processes goes back to Kolmogorov [6], who began the study of their properties. A basic property of a Markov transition function $P_t(x, dy)$ is: the operators T_t given by

$$T_t f(x) = \int f(y) P_t(x, dy)$$

form a *contraction semigroup*, which is in turn associated with a generator and a *resolvent*

$$U^p f(x) = \int_0^\infty e^{-pt} T_t f(x)\, dt.$$

(It is standard in Markov process theory to use P_t for both the transition function and the semigroup of operators.) Thus the whole analytic theory of such semigroups, including the famous Hille-Yosida theory, is available for the study of Markov processes.

Studying a stochastic process *in law* has limitations. Many interesting questions concerning a stochastic process are probabilistic, in the sense that they require reasoning directly with the *sample functions* $t \to X_t$. (Indeed, probabilistic methods occasionally lead to analytic results which have never been proved analytically.

In the case of Markov processes, for example, the structure embodied in its sample paths is much richer than that reflected in its transition function. One can decompose or transform the paths in various ways for which there are no counterparts in the analysis of transition functions.) Two of the pioneers in the study of the sample functions of stochastic processes were Paul Lévy and J. L. Doob, both of whom attacked questions concerning the sample functions of Markov processes, starting in the late 1930s. Doob also did fundamental work on *martingales* and *submartingales*, whose sample path behavior and other properties turned out to be key for many further developments in the theory of stochastic processes in general, and for Markov processes in particular.

In "continuous time" (i.e., when T is an interval of $\mathbf{R}$), sample function analysis demands a considerable amount of measure theory, so during the period starting in 1931 and lasting into the 1950s, Markov processes were studied principally "analytically" (in law) via their transition semigroups and their generators. For example, Feller's important work on the classification of one-dimensional (real-valued) diffusions, performed during this period, was based on analysis of the semigroup. Since Hunt's groundbreaking work [5] in the late 1950s, attention has shifted to the resolvent. Hunt's use of "p-potentials" (U^p) was the forerunner of the modern approach, wherein manipulation of the resolvent replaces reasoning with the semigroup. Hunt is responsible for *many* innovations in both Markov process theory and abstract potential theory; his major contribution was the identification of the two theories.

POTENTIALS

The connection between Brownian motion and Newtonian potential theory, first observed by Kakutani and Kac, was taken up by Doob in a series of papers in the 1950s. This connection springs from the fact that the infinitesimal generator of the Brownian semigroup is a constant times the Laplacian, and it has many manifestations. In particular, harmonic and subharmonic (or superharmonic) functions play a special role for Brownian motion. For instance, let P^x denote the law of Brownian motion B_t in $\mathbf{R}^n$ started at x, and for $a < b$ consider the event Λ that B_t hits the sphere $\{y : |y| = b\}$ before hitting $\{y : |y| = a\}$. It follows from basic properties of Brownian motion that the function $x \to P^x(\Lambda)$ is harmonic in $\{x : a < |x| < b\}$. Doob proved a deeper result in

[4]: when a subharmonic function u is composed with a Brownian motion trajectory B_t, the resulting function $u(B_t)$ is almost surely continuous in t. In the same paper, Doob gives a probabilistic solution of the Dirichlet problem in terms of Brownian paths. It can be shown that virtually all potential theoretic concepts (capacity, equilibrium distribution/potential, energy, etc.) have interpretations in terms of Brownian motion (see [3] or [9]).

Hunt [5] showed that the relationship between the Brownian semigroup and the Newtonian potential kernel was no coincidence. Hunt's theorem states that for a large class of positive kernels V satisfying "the complete maximum principle" of potential theory, there corresponds a contraction resolvent and associated sub-Markovian semigroup P_t, with $Vf = \int_0^\infty P_t f\,dt$. ($V$ is called the "potential kernel" of the semigroup.) Along with this semigroup comes a Markov process which bears the same relationship to the potential theory associated to V as Brownian motion does to Newtonian potential theory (for example, Riesz potential theory corresponds to symmetric stable processes), although some auxiliary hypotheses may be necessary to develop this potential theory fully. Hunt's exposition was the starting point of both a new way to study Markov processes and a new way to look at potential theory. As indicated above, sample function techniques (use of hitting times, transformations, martingale arguments, etc.) are a powerful addition to the arsenal of analytic techniques available to study potential theory; the result is the probabilistic potential theory described in this book.

There is one last point to mention in this brief look at the mathematics behind *Probabilités et Potentiel*. The discussion above, as well as the volume under review, centers on Markov processes. This focus may obscure the synergy (which is more apparent when you read the volumes together) between the general theory of processes, including martingale theory, and Markov process theory. In fact, these theories developed in tandem, each theory at times borrowing ideas or using techniques developed for the other.

This volume

This volume includes two chapters from analytic potential theory (the part having to do with semigroups and resolvents, with resolvents being the fundamental objects). Dellacherie and Meyer describe these chapters as "the development of the analytic parts of Hunt's work," a development which over the past 30 years has been quite extensive. All the basic notions and principles are covered, including the Hille-Yosida theory and Hunt's theorem which concern the construction of resolvents and semigroups. Here too are the ergodic theory for a resolvent, compactification methods (Ray-Knight theory), duality, Dirichlet spaces, and many examples. The next three chapters begin the treatment of Markov processes, presented in what Dellacherie and Meyer call an "elementary form" (primarily for Feller semigroups), to keep technicalities to a minimum. In these chapters the reader meets the probabilistic side of potential theory, via sample function analysis and manipulation. Included are "realizations" of Feller and Ray processes, the strong Markov property, notions associated with first hitting times, the *Lévy system* for dealing with the jumps of the process, and transformations associated with multiplicative functionals. In Chapter XV, one section is devoted to introducing the theory developed to describe the behavior of a Markov process "around" a point of the state space, which brings in the notions of *local time* and *excursions.* In the last chapter, the more general class of *right processes* is discussed briefly (for more on this class of processes, the reader may consult Sharpe's new book [10]).

The authors have promised to continue their discussion of Markov processes in still another volume. If the promises of the preface come true, many of the topics which will appear in the next volume can be found in no other book. Needless to say, the final volume of *Probabilités et Potentiel* is eagerly awaited.

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