

These results, developed by Donsker and Varadhan and to a lesser extent by Gartner, are treated very well in the book. A certain amount of hard analysis is required to handle the ergodicity requirements. These problems of suitable ergodicity conditions for the Markovian case as well as mixing conditions for the non-Markovian case take up the last chapters of the book.

The book contains an extensive list of references as well as detailed historical comments.

Those interested in connections with statistical mechanics should read references [2] or [3].

#### REFERENCES

- [1] H. Cramer, *Sur un nouveau théorème-limite de la théorie des probabilités*, Actualités Sci. Indust. 736 (1938), 5–23. Colloque consacré à la théorie des probabilités, vol. 3, Hermann, Paris.
- [2] R. S. Ellis, *Entropy, large deviations and statistical mechanics*, Springer-Verlag, Berlin, 1985.
- [3] O. E. Lanford, *Entropy and equilibrium states in classical statistical mechanics*, Statistical Mechanics and Mathematical Problems (A. Lenard, ed.), Lecture Notes in Phys. vol. 20, Springer-Verlag, Berlin, 1973, pp. 1–113.
- [4] M. Schilder, *Some asymptotics formulae for Wiener integrals*, Trans. Amer. Math. Soc. 125 (1966), 63–85.

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*Stochastic calculus in manifolds*, by Michel Emery. Springer-Verlag, Berlin, New York, 1989, 151 pp., \$29.00. ISBN 3-540-51664-6

I am glad, but also a little embarrassed to present this book because Emery's work is very closely connected with Paul André Meyer's and mine, these two last ones being also much intertwined. A large part of the book is an exposition of previous work, but also much of the material is new. Anyway, the presentation is always original and interesting. I always prefer intrinsic formulations for manifolds "à la Bourbaki," giving the expression in coordinates

only later, as a tool for proofs or an illustration; Meyer usually goes in the inverse direction. Emery stays in the middle. Each of us tried to help the probabilists absorb stochastic infinitesimal calculus of the second order “without tears”; I don’t know whether any of us succeeded or will succeed. But I guess that Emery’s way will possibly be the best one for that, with his always clear, well-explained, and short statements. This part of probability in differential geometry has become recently more and more important, for instance in large deviations, or in Bismut’s proof of the Atiyah-Singer index theorem.

### §1. PRELIMINARIES ON PROBABILITIES

We shall go fast, not defining everything.  $(\Omega, \mathcal{F}, \mathbf{P})$  is a probability space, equipped with a filtration  $(\mathcal{F}_t)_{t \geq 0}$ , an increasing right-continuous family of sub- $\sigma$ -fields of  $\mathcal{F}$ , indexed by the time  $t$ .  $\mathcal{F}_t$  represents the past and the present at time  $t$ . A process  $X$  is a map  $\mathbf{R}_+ \times \Omega \rightarrow E$ , adapted to the filtration;  $X_t: \omega \mapsto X_t(\omega)$  is the state at the time  $t$ , it should be  $\mathcal{F}_t$ -measurable, and  $X(\omega): t \mapsto X_t(\omega)$  is the orbit defined by  $\omega$ ; a process is a random path on  $E$ . The process  $X$ , if  $E$  is a  $d$ -dimensional vector space, is said to be a semimartingale, if it can be written as a sum  $X = X_0 + A + M$ ,  $X_0$  is the value at time 0,  $A$  a process with (locally) finite variation,  $M$  a (local) martingale;  $A$  is a signal,  $M$  a noise.  $A$  is called the compensator of  $X$  and denoted  $\tilde{X}$ ,  $M$  the compensated and denoted  $X^c$ ; they are unique, up to a set of  $\mathbf{P}$ -measure 0 (P. A. Meyer) if  $X, A, M$  are continuous (a.e.), which will be always assumed. The word “local” will always be omitted.

K. Ito introduced the stochastic integration with respect to a martingale; P. A. Meyer introduced the semimartingales to extend it to them; it is

$$(1.1) \quad J = H \cdot X, \quad J_t = \int_0^t H_s dX_s$$

( $\omega$  is always omitted), where  $H$  is an optional (some criterion of measurability), locally bounded process,  $\mathcal{L}(E; F)$ -valued ( $F$  another vector space), and then  $J$  is  $F$ -valued;  $J$  is a new semimartingale and is of finite variation or a martingale if  $X$  is<sup>1</sup>.

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<sup>1</sup> All these results can be found in the appendix of P. A. Meyer at the end of Emery’s book, with also further references.

A martingale is so oscillatory that it has no finite variation, so also  $X$  in general; but it has a quadratic variation (coming from the martingale part only), and a 0 cubic variation. This defines the bracket:

$$[X, X]_t = X_0^2 + \lim_{|\Delta| \rightarrow 0} \sum_i (X_{t_{i+1} \wedge t} - X_{t_i \wedge t})^2,$$

the square being taken in the tensor product  $E \otimes E$  or the symmetric tensor product  $E \odot E$  (factor space of  $E \otimes E$ ), then  $[X, X]$  too; the limit, as the size  $|\Delta|$  of the subdivision  $\Delta = (t_0 = 0, t_1, t_2, \dots)$  converges to 0, is a limit in probability, uniformly for  $t$  in any compact set of  $\mathbf{R}_+$ .

At the end of Emery's book there is an appendix by P. A. Meyer, giving the bases of probabilities.

Later on, Ito's differentiation formula is the following: If  $\Phi$  is a  $C^2$  map  $E \rightarrow F$ , vector spaces,  $X$  and  $E$ -semimartingale, then  $\Phi(X)$  is an  $F$ -semimartingale, with the integral formula:

$$(1.2) \quad \Phi(X_t) - \Phi(X_0) = \int_0^t \Phi'_s(X) dX_s + \frac{1}{2} \int_0^t \Phi''(X_s) d[X, X]_s.$$

The occurrence of the second derivative is due to the quadratic variation of  $X$ , making necessary a Taylor formula of the second order. One may write (1.1) and (1.2) in differential expressions:

$$(1.3) \quad dJ_t = H_t dX_t$$

$$(1.4) \quad d(\Phi(X))_t = \Phi'(X_t) dX_t + \frac{1}{2} \Phi''(X_t) dX_t dX_t,$$

from which we deduce

$$(1.5) \quad d(\Phi(X))_t d(\Phi(X))_t = \Phi'(X_t) \odot \Phi'(X_t) dX_t dX_t,$$

where  $dX_t dX_t$  or  $dX_t \otimes dX_t$  or  $dX_t \odot dX_t = d[X, X]_t$  is adapted to the definition of the bracket as quadratic variation. Emery doesn't use differential expressions much which I personally prefer; let me use them here systematically.

Now one comes to stochastic differential equations (SDE):

$$(1.6) \quad dX_t = \sum_{k=1}^m H_k(X_t) dZ_t^k, \quad X_0 = x,$$

meaning

$$(1.7) \quad X_t = x + \sum_{k=1}^m \int_0^t H_k(X_s) dZ_s^k.$$

Here the  $Z^k$  are given real semimartingales,  $H_k$  given vector fields on  $E$ ,  $x$  given the initial value, and  $X$  is the unknown,  $E$ -valued semimartingale. One may abbreviate that by

$$(1.8) \quad dX_t = H(X_t) dZ_t, \quad X_0 = x,$$

$Z$  being a given  $G$ -valued semimartingale,  $G$  a vector space, and  $H$  an  $\mathcal{L}(G; E)$ -valued vector field on  $E$ . The field  $H$  has to be locally Lipschitz. Then the equation has one and only one solution a.e., with a death-time  $\zeta$ , random variable  $\omega \mapsto \zeta(\omega) \leq +\infty$ ; on  $\{\zeta < +\infty\}$ ,  $X_t$  converges to infinity in  $E$  if  $t$  converges to  $\zeta$ . Emery writes only that  $X([0, \zeta])$  is not relatively compact, but specifies that  $X_t$  may or may not converge to infinity for  $t \rightarrow \zeta < +\infty$  (page 87); actually it converges to infinity, I proved it myself.<sup>2</sup>

## §2. PRELIMINARIES ON DIFFERENTIAL GEOMETRY

The vector spaces are replaced by  $C^\infty$  manifolds. Let  $T_x(M)$  denote the tangent vector space at the point  $x$  of the  $d$ -dimensional manifold  $M$ ,  $T(M)$  the tangent vector bundle;  $T_x^*(M)$ , the dual of  $T_x(M)$ , is the cotangent space,  $T^*(M)$  the cotangent vector bundle. A  $C^\infty$  field of tangent vectors, or a Lie field, is a  $C^\infty$  differential operator of order 1, without term of order 0; a field of cotangent vectors is a differential form of degree 1.

A bilinear form ( $C^\infty$ ) will be a section  $b$  of the vector bundle  $(T(M) \otimes T(M))^*$ . All these  $C^\infty$  fields are  $C^\infty$  modules, finitely generated (using the embedding property of Whitney, of  $M$  into  $\mathbf{R}^{2d}$ ). In a chart,  $M = E$ , vector space,  $T_x(M) = E$ ,  $T_x^*(M) = E^*$ , a bilinear form at  $x$  is a bilinear form on  $E \times E$  or a linear form on  $E \otimes E$ . Therefore,  $T_x(M)$  and  $T_x^*(M)$  have dimension  $d$ , as does  $M$ . With coordinates,  $\xi \in E$ ,  $\xi = \sum_k b^k D_k$ ;  $\xi^* \in E^*$ ,  $\xi^* = \sum_k b_k^* dx^k$ ; then  $\langle \xi^*, \xi \rangle_{E^*, E} = \sum_k b_k^* b^k$ . If  $\varphi$  is a  $C^1$  real function on  $E$ ,  $D^1\varphi(x) = \varphi'(x) = \sum_k D_k\varphi(x) dx^k$ ; for  $\xi \in E$ ,  $(\xi\varphi)(x) = \langle D^1\varphi(x), \xi \rangle = \sum_k b^k D_k\varphi(x)$ .

It's necessary, for the infinitesimal stochastic calculus, to introduce also the second-order fields or covector fields. The space of second derivatives at  $x \in M$  will be denoted by  $\tau_x(M)$ ,  $\tau(M)$  will be the second-order tangent bundle. A second-order tangent field will be a differential operator of order at most 2, without term of degree 0. Similarly,  $\tau_x^*(M)$  will be the dual of  $\tau_x(M)$ ,  $\tau^*(M)$  the corresponding second-order cotangent bundle.

<sup>2</sup> [46] in the bibliographical index of Emery's book, Proposition 7.4, page 97.

If  $M = E$ ,  $L \in \tau_x(E)$ , then

$$L = \sum_{k=1}^d b^k D_k + \sum_{i,j=1}^d a^{i,j} D_i D_j, \quad a^{i,j} = a^{j,i}.$$

It is often convenient to write an element of  $\tau_x(E)$  by a vertical matrix,

$$L = \begin{pmatrix} \sum_k \beta^k D_k \\ \sum_{i,j} a^{i,j} D_i D_j \end{pmatrix} \in \begin{pmatrix} E \\ \oplus \\ E \odot E \end{pmatrix};$$

therefore,  $\tau_x(E)$  and also  $\tau_x(M)$  have dimension  $d + d(d+1)/2$ . An element of  $\tau_x^*(E)$  will be written as a horizontal matrix:

$$L^* = \sum_k b_k^* dx^k + \sum_{i,j} a_{i,j}^* dx^i dx^j, \quad a_{i,j} = a_{j,i}$$

or

$$L^* = \left( \begin{array}{cc} \sum_k b_k^* dx^k & \sum_{i,j} a_{i,j}^* dx^i dx^j \end{array} \right);$$

the second term is not an exterior form of degree 2, but a bilinear symmetric form on  $E \times E$ , or a linear form on  $E \odot E$ .

Then  $\tau_x^*(E) = E^* \oplus (E \odot E)^*$  or, as a horizontal matrix,  $(E^* \oplus (E \odot E)^*)$ ; it also has dimension  $d + d(d+1)/2$ , and  $\tau_x^*(M)$  does too.

If  $\varphi$  is a  $C^2$  real function on  $E$ ,  $D^2\varphi(x) = \varphi'(x) + \varphi''(x)$ , or  $(\varphi'(x) \ \varphi''(x))$  (horizontal matrix); for  $L \in E \oplus (E \odot E)$ ,

$$L\varphi(x) = \langle D^2\varphi(x), L \rangle = \sum_k b^k D_k \varphi(x) + \sum_{i,j} a^{i,j} D_i D_j \varphi(x).$$

We keep  $(E \odot E)^*$ , and don't identify it with  $E^* \otimes E^*$ ; it has the advantage of cancelling a lot of factors  $1/2$ . A symmetric bilinear form will be considered as a bilinear form, that is, an element of  $E^* \otimes E^*$ , which happens to be symmetric. Thus  $dx^i dx^j$  is  $dx^i \otimes dx^j$ , and is not symmetric, but

$$\sum_{i,j} a_{i,j}^* dx^i dx^j \text{ is if } a_{i,j}^* = a_{j,i}^*.$$

On a manifold  $M$ , it is no longer true that  $\tau_x(M) = T_x(M) \oplus (T_x(M) \odot T_x(M))$ , but  $T_x(M) \subset \tau_x(M)$ , and the factor space  $\tau_x(M)/T_x(M)$  is canonically isomorphic to  $T_x(M) \odot T_x(M)$ . One has the exact sequence

$$(2.0) \quad 0 \rightarrow T_x(M) \rightarrow \tau_x(M) \rightarrow T_x(M) \odot T_x(M) \rightarrow 0,$$

which splits if  $M = E$ , a vector space. On the duals,  $T_x^*(M)$  is the factor space of  $\tau_x^*(M)$  by the orthogonal  $T_x(M)^+$  of  $T_x(M)$ ; one has the exact sequence

$$(2.1) \quad 0 \rightarrow T_x(M)^+ \rightarrow \tau_x^*(M) \rightarrow T_x^*(M) \rightarrow 0,$$

with  $T_x(M)^+ = (T_x(M) \odot T_x(M))^*$ ; it also splits if  $M = E$ ; the image of  $D^2$  in  $(E \odot E)^*$  is called the Hessian of  $\varphi$ ,  $\text{Hess } \varphi = \text{Hess}(D^2\varphi) = \sum_{i,j} D_i D_j \varphi dx^i dx^j$ .

If  $\Phi$  is a  $C^2$  map from  $M$  into another manifold  $N$ , it has tangent maps,  $T_x\Phi : T_x(M) \rightarrow T_{\Phi(x)}N$  and  $\tau_x\Phi : \tau_x(M) \rightarrow \tau_{\Phi(x)}N$ ;  $\tau_x\Phi$  induces  $T_x\Phi$  on  $T_x(M)$ , therefore there is a factor map  $\tau_x/T_x$ :

$\tau_x(M)/T_x(M) \rightarrow \tau_{\Phi(x)}(N)/T_{\Phi(x)}(N)$ ; with respect to the above exact sequence, it is exactly the map

$$T_x\Phi \odot T_x\Phi : T_x(M) \odot T_x(M) \rightarrow T_{\Phi(x)}N \odot T_{\Phi(x)}N.$$

In other words, the following diagram is commutative:

$$\begin{array}{ccccccc} 0 \rightarrow & T_x(M) & \rightarrow & \tau_x(M) & \rightarrow & T_x(M) \odot T_x(M) & \rightarrow 0 \\ & \downarrow T_x\Phi & & \downarrow \tau_x\Phi & & \downarrow T_x\Phi \odot T_x\Phi & \\ 0 \rightarrow & T_{\Phi(x)}N & \rightarrow & \tau_{\Phi(x)}(N) & \rightarrow & T_{\Phi(x)}N \odot T_{\Phi(x)}N & \rightarrow 0. \end{array}$$

Such morphisms between second-order tangent spaces are called by Emery Schwartz morphisms; I am not the author of this terminology, of course; such morphisms are known among differential geometers, but it is true that I was responsible for their introduction and systematic use in stochastic infinitesimal calculus. Using charts, if  $\Phi$  is  $C^2 : E \rightarrow F$ , and if we write elements of  $\tau_x$  as vertical matrices,  $T_x(\Phi)$  is the derivative  $\Phi'(x)$ , and  $\tau_x(\Phi)$  is a square matrix

$$(2.2) \quad \tau_x\Phi = \begin{pmatrix} \Phi'(x) & \Phi''(x) \\ 0 & \Phi'(x) \odot \Phi'(x) \end{pmatrix},$$

a Schwartz morphism

$$\tau_x(E) = \begin{pmatrix} E \\ \oplus \\ E \odot E \end{pmatrix} \rightarrow \tau_{\Phi(x)}F = \begin{pmatrix} F \\ \oplus \\ F \odot F \end{pmatrix},$$

$\Phi''(x)$  is the usual second derivative, symmetric bilinear form  $E \times E \rightarrow F$ , or linear map  $E \odot E \rightarrow F$ ;  $\Phi'(x) \odot \Phi'(x)$  is the square tensor map  $E \odot E \rightarrow F \odot F$  of  $\Phi'(x) : E \rightarrow F$ .

§3. SEMIMARTINGALES ON MANIFOLDS,  
AND THEIR DIFFERENTIAL CALCULUS

Although Brownian motions of manifolds have been used decades ago by Ito and other probabilists, curiously enough the notion of semimartingales on manifolds had not been defined; I introduced them and studied them only in 1980, and many results on them were given by P. A. Meyer and me, and many others. See references [3], [4], [5], [8], [13], [15], [16], [19], [34], [35], [36], [43], [44], [45], [46], [52].

A stochastic process  $X : \mathbf{R}_+ \times \Omega \rightarrow M$  is said to be a semimartingale if, for every real  $C^2$  function  $\varphi$ ,  $\varphi(X)$  is a real semimartingale. Consequently, if  $\Phi$  is a  $C^2$ -map  $M \rightarrow N$ ,  $\Phi(X)$  will be a semimartingale on  $N$ . Of course, a decomposition as  $X = X_0 + A + M$  doesn't make sense, nor does the notion of a martingale on a manifold. But differentials of semimartingales will make sense. For the above map  $\Phi : M \rightarrow N$ , if we take charts  $M \simeq N$ ,  $N \simeq F$ , (1.4) and (1.5) will be written, according to (2.2):

$$(3.1) \quad \begin{pmatrix} d(\Phi(X))_t \\ \frac{1}{2}d(\Phi(X))_t d(\Phi(X))_t \end{pmatrix} = \begin{pmatrix} \Phi'(X_t) & \Phi''(X_t) \\ 0 & \Phi'(X_t) \odot \Phi'(X_t) \end{pmatrix} \begin{pmatrix} dX_t \\ \frac{1}{2}dX_t dX_t \end{pmatrix};$$

or, if we put

$$dX_t = \begin{pmatrix} dX_t \\ \frac{1}{2}dX_t dX_t \end{pmatrix} \in \begin{pmatrix} E \\ \oplus \\ E \odot E \end{pmatrix}, \quad d(\Phi(X))_t \in \begin{pmatrix} F \\ \oplus \\ F \odot F \end{pmatrix},$$

then

$$d(\Phi(X))_t = \tau_{X_t} \Phi \quad dX_t.$$

In coordinates, if  $X = \sum_k X^k D_k$ ,

$$dX_t = \sum_k X^k D_k + \frac{1}{2} \sum_{i,j} dX^i dX^j D_i D_j.$$

This becomes intrinsic and goes from the chart on  $E$  to the manifold  $M$  itself: to the semimartingale  $X$  on  $M$ , and the point  $x \in M$ , we may affect  $dX_t$ , differential of  $X$  at  $X_t$ , a "small element of semimartingale" at  $X_t$ , element of  $\tau_{X_t}(M)$ ; there is no differential  $dX_t \in T_{X_t}(M)$  (except if  $X$  has finite variation). This notion is rather sophisticated; Emery says humorously: "should the differential exist, it would have the geometrical nature of a

second order tangent vector; if you do not (or not yet) believe in  $\mathbf{d}X_t$ , the statement is vacuously satisfied.”

While the decomposition  $X = X_0 + \tilde{X} + X^c$  doesn't make sense, one may write

$$(3.2) \quad \begin{cases} \mathbf{d}X = \mathbf{d}\tilde{X} + dX^c \\ \mathbf{d}X_t \in \tau_{X_t}(M), \mathbf{d}\tilde{X}_t \in \tau_{X_t}M, dX_t^c \in T_{X_t}(M), \end{cases}$$

and the common image of  $\mathbf{d}X_t$  and  $\mathbf{d}\tilde{X}_t$  on  $T_t(M) \otimes T_t(M)$  by the canonical projection

$$\pi : \tau_t(M) \rightarrow \tau_t(M)/T_t(M) = T_t(M) \otimes T_t(M)$$

is  $\frac{1}{2}dX_t \otimes dX_t$  (although  $dX_t$  doesn't exist), it behaves under  $\Phi$  as

$$\begin{aligned} \frac{1}{2}d(\Phi(X))_t d(\Phi(X))_t &= \tau_t/T_t(\Phi) \frac{1}{2}dX_t dX_t \\ &= (T_t(\Phi) \otimes T_t(\Phi)) \frac{1}{2}dX_t dX_t. \end{aligned}$$

One may say that  $dX_t^c$  is the martingale component of  $\mathbf{d}X_t$ ,  $\mathbf{d}\tilde{X}_t$  its finite variation component, and that  $dX_t dX_t$  is the bracket of  $\mathbf{d}X_t$ . All of that appears clearly in charts.

More rigorous definitions of  $\mathbf{d}X_t$ ,  $\mathbf{d}\tilde{X}_t$ ,  $dX_t^c$ ,  $\frac{1}{2}dX_t dX_t$  should be given, though the previous intuitive approximations are sufficient never to make mistakes in the applications. I gave this definition, using the notion of differentials of semimartingales, sections on vector bundles on  $\mathbf{R}^+ \times \Omega$ ; <sup>3</sup>

Michel Emery knows that very well, but he resigned himself to define them in this way; it would have been too long here for a limited benefit; in deeper studies, it is indispensable.

One may perform integrations with respect to these differentials. If  $\Theta$  is an optional locally bounded second-order form over  $X$  on  $M$  (not a differential form of order 2 which would be an exterior form, with an antisymmetric property), that is,  $\Theta_t(\omega) \in \tau_{X_t(\omega)}^*(M)$ , then formally  $\langle \Theta(X_t), \mathbf{d}X_t \rangle$ , the scalar product, is a small real number, and one has a stochastic integral  $t \mapsto \int_0^t \langle \Theta(X_s), \mathbf{d}X_s \rangle$  which is a real semimartingale. These integrals are carefully studied in Chapter VI; there exists also  $\int \langle \Theta, \mathbf{d}\tilde{X} \rangle$ , a real process with finite variation,  $\int \langle \theta, dX^c \rangle$ , a martingale, if  $\theta$  is a differential form of degree 1, and  $\int \langle b, dX dX \rangle$  a process with

<sup>3</sup> [46] in the bibliography of Emery's book.



finite variation if  $b$  is a bilinear form (see beginning of §2); for a bilinear form which is not symmetric, we must consider  $dX_t dX_t$  as an element of  $T_x(M) \otimes T_x(M)$  instead of  $\odot$ , or replace  $b$  by its associated symmetric form. If  $T = D^2\varphi$ ,  $\varphi$  a real  $C^2$  function on  $M$ , then  $\int_0^t \langle D^2\varphi(X_t), dX_t \rangle = \varphi(X_t) - \varphi(X_0)$ .

It is to be noted that Emery goes from elementary objects to more sophisticated ones (differential calculus “without tears”); he studies  $\int \langle b, dXdX \rangle$  early in the book, Chapter III, but  $\int \langle \Theta, dX \rangle$  only in Chapter VI. He makes an extensive study of the integrals of bilinear forms;  $\int \langle b, dXdX \rangle$  is the  $b$ -quadratic variation of  $X$ . It can be computed by a discretization and a limiting procedure. Emery goes still further in pushing  $dX$  far in the book: He studies in Chapter IV the semimartingales and connections, and in Chapter V the Brownian motion on manifolds, all before the definition of  $dX$ . We explain them here in the inverse order, starting from Chapter VI; it doesn't deform his work at all.

§4. CONNECTIONS ON MANIFOLDS,  
MARTINGALES WITH RESPECT TO A CONNECTION

It is known that a linear connection with zero torsion on the fiber bundle  $T(M)$  is equivalent to a splitting of the exact sequence (2.1); it decomposes  $\tau(M)$  as a direct sum

$$(4.1) \quad \tau(M) = T(M) \oplus H(M),$$

where the projection of  $H(M)$  on  $T(M) \odot T(M)$  is bijective. We shall write it as

$$(4.2) \quad \tau(M) = T(M) \oplus \overline{T(M) \odot T(M)},$$

$T(M)$  will be called the vertical subspace of  $\tau(M)$ ,  $\overline{T(M) \odot T(M)}$ , the horizontal subspace.

One has a corresponding decomposition of the dual space  $\tau^*M$ :

$$(4.3) \quad \tau^*(M) = \overline{T^*(M)} \oplus (T(M) \odot T(M))^*.$$

Without any connection,  $T(M)$  and  $T(M)^+ = (T(M) \odot T(M))^*$  are always subspaces, but here  $\overline{T(M) \odot T(M)}$  and  $\overline{T^*(M)}$  too, instead of  $T(M) \odot T(M)$ ,  $T^*(M)$ , factor spaces. Emery defines the connection by the operator Hess, as a generalization of Hess of (2.1) which corresponds to the trivial connection on  $E$ : Hess is the projection of  $\tau^*(M)$  on  $T(M)^+ = (T(M) \odot T(M))^*$ .

On a chart, the connection is defined by  $\Gamma$ ,  $\Gamma(x)$  being a symmetric bilinear map:  $T_x(M) \times T_x(M) \rightarrow T_x(M)$ , or a linear map

$$T_x(M) \odot T_x(M) \rightarrow T_x(M);$$

$$\Gamma(x) = \sum_{i,j,k=1}^d \Gamma_{i,j}^k(x) dx^i dx^j D_k.$$

The decomposition of an element  $\begin{pmatrix} u \\ v \end{pmatrix}$  of

$$\begin{pmatrix} E \\ \oplus \\ E \odot E \end{pmatrix}$$

as a sum of elements of  $T_x(M)$  and  $\overline{T_x(M) \odot T_x(M)}$  is

$$(4.4) \quad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u + \Gamma(x)v \\ 0 \end{pmatrix} + \begin{pmatrix} -\Gamma(x)v \\ v \end{pmatrix}.$$

$$(4.5) \quad \text{Therefore } \begin{pmatrix} u \\ v \end{pmatrix} \text{ is horizontal iff } u + \Gamma(x)v = 0.$$

The lifting

$$E \odot E \rightarrow \begin{pmatrix} E \\ \oplus \\ E \odot E \end{pmatrix}$$

is  $v \mapsto \begin{pmatrix} -\Gamma(x)v \\ v \end{pmatrix}$ ,  $-\Gamma(x)$  is the component of this lifting on  $E$ .

The decomposition of  $(\alpha \ \beta) \in (E^* \oplus (E \odot E)^*)$  is

$$(4.6) \quad (\alpha \ \beta) = (\alpha \ \alpha \circ \Gamma(x)) + (0 \ -\alpha \circ \Gamma(x) + \beta).$$

Then

$$(4.7) \quad \text{Hess}_\Gamma \varphi = \text{Hess}_\Gamma(\varphi' \ \varphi'') = -\varphi' \circ \Gamma + \varphi'' \in (E \odot E)^*.$$

One can learn the main properties of connections, geodesics, and the Levi-Civita connection by these notations.

The decomposition of the differential  $dX$  of a semimartingale in (3.1) has an analogy when there is a connection; one may write

$$(4.8) \quad dX = \overline{dX} + \frac{1}{2} \overline{dX dX},$$

$$\overline{dX}_t \in T_{X_t}(M), \frac{1}{2} \overline{dX}_t dX_t \in T_{X_t}(M) \odot T_{X_t}(M);$$

$$\frac{1}{2} \overline{dX}_t dX_t \in \overline{T_{X_t}(M) \odot T_{X_t}(M)}$$

here there exists a  $\overline{dX}_t \in T_{X_t}(M)$ , and  $\frac{1}{2} \overline{dX}_t dX_t$  lies in  $\tau_{X_t}(M)$  while  $\frac{1}{2} dX_t dX_t$  lies in the quotient  $T_{X_t}(M) \odot T_{X_t}(M)$ . We write  $\overline{dX}_t$  instead of  $dX_t$  since it exists only because of the connection.

Let  $\gamma$  be a smooth curve on  $M$ ; the speed  $\dot{\gamma}_t$  is the image of  $1 \in \mathbf{R}$  in  $T_{\gamma_t}(M)$  through  $T_t\gamma$ , and the acceleration  $\ddot{\gamma}_t$  is the image of  $1 \odot 1 \in \mathbf{R} \odot \mathbf{R} \subset \mathbf{R} \oplus (\mathbf{R} \odot \mathbf{R})$  in  $\tau_{\gamma_t}(M)$  through  $\tau_t$  by (2.2):

$$(4.9) \quad \ddot{\gamma} = \begin{pmatrix} \gamma_t'' \\ \gamma_t' \odot \gamma_t' \end{pmatrix} \in \begin{pmatrix} E \\ \oplus \\ E \odot E \end{pmatrix} \text{ on a chart.}$$

On  $E$  equipped with the trivial connection,  $\gamma$  is a geodesic if it is a straight line, i.e.  $\gamma'' = 0$ ; therefore one defines a geodesic of the connection as a curve whose acceleration  $\ddot{\gamma}$  is horizontal, which is written on the chart, according to (4.5):

$$(4.10) \quad \gamma'' + \Gamma(\gamma)(\gamma', \gamma') = 0.$$

It is well known!

A function  $\varphi$  on  $E$  is convex if  $\varphi'' \geq 0$ . It means that the component of  $D^2\varphi = (\gamma' \gamma'')$  on  $(E \odot E)^*$  is  $\varphi'' \geq 0$ . Therefore one says naturally that a function  $\varphi$  on  $M$  equipped with a connection is convex if the component of  $D^2(\varphi)$  on  $(T(M) \odot T(M))^* = T(M)^+$  is  $\geq 0$ , that is, if  $\text{Hess } \varphi \geq 0$  (Hess of the connection!).

Finally, using (4.7) and comparing with the case of  $E$  with its trivial connection, a semimartingale  $X$  on  $M$  will be said to be a martingale with respect to the connection if the vertical component  $\overline{dX}$  of  $\underline{dX}$  on  $T_X(M)$  is a martingale, or if the component  $\widetilde{dX}$  of  $\underline{dX}$  is zero, and, according to (4.5), it is written, in a chart

$$(4.11) \quad d\widetilde{X}_t + \Gamma(X_t) \frac{1}{2} dX_t dX_t = 0$$

or

$$(4.12) \quad d\widetilde{X}_t^k + \Gamma_{i,j}^k(X_t) \frac{1}{2} dX_t^i dX_t^j = 0 \quad \forall k.$$

This enables Emery to describe elegantly the main results of Emery, Zheng, Duncan, Darling, P. A. Meyer, and myself, on the relationships between geodesics, convex functions, and martingales, in a connection; also the convergence of martingales at infinity ( $t \rightarrow +\infty$ ); everything is going well and smoothly. Precise references are given at the end of Chapter IV of Emery, with the publications of the author at the bibliographical index page 125 of the book.

Observe that I mentioned here  $\underline{dX}$ ; it arises only in Chapter VI, while the previous results constitute Chapter IV. Then they have

to be expressed without the notation  $dX$ , which is easy. It will be the same for the Brownian motion in Chapter V. It corresponds to the intention to introduce  $dX$ , probably the best tool in all of that, but rather sophisticated, as late as possible, after it has been used indirectly many times, so that it falls as a ripe fruit.

§5. THE BROWNIAN MOTION AND THE LEVI-CIVITA CONNECTION (CHAPTER V)

The Riemannian structure is defined by the fundamental quadratic form  $g \in (T(M) \odot T(M))^*$ ,  $g = \sum_{i,j} g_{i,j} dx^i dx^j$  on a chart. The Levi-Civita connection is defined by its Hess,

$$(5.1) \quad \text{Hess } \varphi = \frac{1}{2} \mathcal{L}_{\text{grad } \varphi} g \in (T(M) \odot T(M))^*,$$

$\mathcal{L}$  being the Lie derivative.

As it is usual now, a Brownian motion  $X$  on  $M$ , over a probability space  $(\Omega, \mathcal{F}, \mathbf{P}, (\mathcal{F}_t))$  is a semimartingale satisfying the following “problem of martingales”:

$$(5.2) \quad \forall \varphi \in C^2 \text{ real, } t \mapsto \varphi(X_t) - \varphi(X_0) - \int_0^t \frac{1}{2} \Delta \varphi(X_s) ds$$

is a real martingale,  $\Delta$  is the Laplacian; one says also that  $X$  is a  $\frac{1}{2}\Delta$ -diffusion.

As usual, this integral condition can be written in a differential form:  $\Delta$  is a second-order differential operator, therefore, a section of the fiber bundle  $\tau(M)$ ; (5.2) is equivalent to:

$$(5.3) \quad d\tilde{X}_t = \frac{1}{2} \Delta(X_t) dt$$

$$\Delta(X_t) \in \tau_{X_t}(M).$$

The various known expressions of  $\Delta$  show that it is a differential operator of pure order 2, or a horizontal derivative,  $\Delta \in \overline{T(M) \odot T(M)}$ , with respect to the connection; that is, its component on  $T(M)$  according to (4.2) is 0. Therefore,  $d\tilde{X}_t$  is horizontal too; the Brownian motion is a martingale with respect to the connection. Emery gives various interesting properties, but of course doesn't prove here the existence and uniqueness (in law) of it; this can be done only by solving a SDE, which is the subject of Chapter VI.

§6. STOCHASTIC DIFFERENTIAL EQUATIONS (SDE)

Chapter VI defines  $dX$  and will be able to put an SDE on a manifold in an intrinsic form. An equation such as (1.8) doesn't

make sense, since  $dX$  doesn't exist on  $M$ , only  $\mathbf{d}X$  exists. On a chart,  $M \simeq E$ , (1.8) is written

$$(6.1) \quad \begin{cases} dX_t = H(X_t) dZ_t \\ \frac{1}{2} dX_t dX_t = H(X_t) \odot H(X_t) \frac{1}{2} dZ_t dZ_t, \end{cases}$$

where  $[Z, Z]$  takes its values in  $G \odot G$ , and  $H \odot H$  is the square tensor map  $G \odot G \rightarrow E \odot E$  of  $H : G \rightarrow E$ .

Therefore  $dZ dZ$  necessarily occurs.

Let us consider more generally the equation

$$(6.2) \quad dX_t = H(X_t) dZ_t + K(X_t) \frac{1}{2} dZ_t dZ_t,$$

where  $H$  is a field of  $\mathcal{L}(G; E)$ -vectors, and  $K$  a field of  $\mathcal{L}(G \odot G; E)$ -vectors. Then

$$(6.3) \quad \mathbf{d}X_t = \begin{pmatrix} H(X_t) & K(X_t) \\ 0 & H(X_t) \odot H(X_t) \end{pmatrix} dZ_t.$$

This square matrix defines exactly a Schwartz morphism from  $G \oplus (G \odot G)$  into  $E \oplus (E \odot E)$ . Therefore, one can go to manifolds, and define a SDE on  $M$  as:

$$(6.4) \quad \mathbf{d}X = f(X) \mathbf{d}Z,$$

where  $dZ = (\frac{1}{2} \frac{dZ}{dZ dZ})$ ;  $Z$  is a given  $G$ -valued semimartingale, and  $f$  is a given field of Schwartz morphisms,  $f(x)$  is a Schwartz morphism from  $\tau_x G = G \oplus (G \odot G)$  into  $\tau_x(M)$ .

Michel Emery gives a still better generalization which will surely prove very fruitful. He replaces  $G$  also by a manifold.

Let  $M, N$ , be  $C^\infty$ -manifolds, and  $f$  a field of Schwartz  $(\tau(M), \tau(N))$ -morphisms on  $M \times N$ : for  $(x, y) \in M \times N$ ,  $f(x, y)$  is a Schwartz morphism  $\tau_x(M) \rightarrow \tau_y(N)$ .

Then if  $X$  is a given  $M$ -valued semimartingale,  $Y$  the unknown semimartingale on  $N$ ,

$$(6.5) \quad \mathbf{d}Y = f(X, Y) \mathbf{d}X,$$

$Y_0 = y \in N$ , is a SDE on  $N$  in the most general form, every vector space has disappeared. If  $f$  is continuous and locally Lipschitz with respect to the second variable, there is one and only one solution with a death time  $\zeta$ , and the usual properties.

### §7. STRATONOVITCH AND ITO INTEGRALS

Assume, coming back to §1, that  $H$  is not only optional, but more regular, i.e. a  $C^1$  function of a semimartingale. Then its

Stratonovitch integral is defined by

$$(7.1) \quad \int_0^t H_s \delta X_s = \int_0^t H_s dX_s + \frac{1}{2} dH_s dX_s.$$

One may give another definition, using a notion of differentiation of forms of degree 1 into forms of degree 2, introduced and studied by P. A. Meyer; these forms of degree 2 are not at all exterior forms, but sections of  $(T(M) \otimes T(M))^*$  (neither antisymmetric nor symmetric) as described already in §2; Emery makes a systematic use of this notion in this chapter. One of the great interests of the Stratonovitch integrals is that they don't use the second-order tangent vectors. The Ito formula is replaced by the Stratonovitch formula for a function  $\Phi$  of class  $C^2$  and a semimartingale  $X$ :

$$(7.2) \quad \delta\Phi(X) = \Phi'(X)\delta X,$$

analogous to the usual transform when  $X$  has a finite variation.

Therefore always second-order tangent or cotangent vectors of the second order will be replaced by the analogous ones of the first order. But, of course, there are some inconveniences. For instance, if  $X$  is a martingale,  $\int H \delta X$  need not be a martingale; the Stratonovitch integral doesn't allow us to distinguish the martingales among the semimartingales; one must always go to Ito integrals for that. Also one cannot handle directly the relationship with the connection, and a more complicated way is necessary. But another interest of the Stratonovitch formalism is the transfer principle of Malliavin (an intuitive one): "every" property which is true for smooth curve extends to semimartingales, using Stratonovitch integrals. For instance, nice discretizations exist for the computation of integrals or of solutions of SDE.

On a manifold  $M$ , the Stratonovitch differential  $\delta X_t$  of a semimartingale  $X$  exists, and it is a first-order tangent vector at the point  $X_t$ ,  $\delta X_t \in T_{X_t}(M)$ . If  $\theta$  is a process of first-order cotangent vectors above  $X$ ,  $\theta(t, \omega) \in T_{X_t(\omega)}^*(M)$ , regular enough, then the integral  $\int \langle \theta, \delta X \rangle$  exists.

Finally, one may define a SDE on  $N$  by the same way as (6.5), but with first-order tangent vectors. If  $e$  is a field of  $(T(M), T(N))$ -morphisms on  $M, N$ ,  $e(x, y) \in \mathcal{L}(T_x(M), T_y(N))$ , of class  $C^1$ -Lipschitz and if  $X$  is a given semimartingale on  $X$ ,

then the following formula is a Stratonovitch SDE for  $Y$  on  $N$ :

$$(7.3) \quad \delta Y = e(X, Y)\delta X, Y_0 = y;$$

it has a unique solution, with a death time  $\zeta$ .

There is the very interesting property: there exists one and only one  $f$  as in (6.5),  $f(x, y)$  Schwartz morphism  $\tau_x(M) \rightarrow \tau_y(N)$ , having the following property: if  $x$  and  $y$  are curves on  $M, N$  respectively, of class  $C^2$ , such that  $\dot{y} = e(x, y)\dot{x}$ , then the accelerations satisfy  $\ddot{y} = f(x, y)\ddot{x}$ . Then (7.3) is equivalent to

$$(7.4) \quad dY = f(X, Y) dX.$$

Thus Stratonovitch SDE is led back to Ito SDE.

There exists another way, completely different from the Stratonovitch one, to write SDE only with tangent vectors. Put on  $M, N$ , two arbitrary connections with no torsion. First of all, if  $\Theta$  is an optional process cotangent of second order above  $X$ ,  $f(\Theta, dX)$  exists according to the end of our §3, also the differential  $\langle \Theta, dX \rangle$ ; therefore  $\langle \Theta, \overline{dX} \rangle$  and  $\Theta \frac{1}{2} \overline{dX} d\overline{X}$  exist also, for instance  $\langle \Theta, \overline{dX} \rangle = \langle \Theta, dX + \Gamma(X) \frac{1}{2} dX d\overline{X} \rangle$  on a chart; but, if  $\theta$  is an optional process of order 1 above  $X$ , it is a fortiori of the second order because, with the connection,  $T_X^*(M) \simeq \overline{T_X^*(M)} \subset \tau_X^*(M)$ ; therefore  $\langle \theta, \overline{dX} \rangle$  exists too, which is normal because  $T_X^*(M)$  is the dual of  $T_X(M)$  and  $\overline{dX} \in T_X(M)$ :

$$(7.5) \quad \begin{aligned} \langle \theta, \overline{dX} \rangle &= (\theta \quad \theta \cdot \Gamma(X)) \begin{pmatrix} dX + \frac{1}{2} \Gamma(X) dX d\overline{X} \\ 0 \end{pmatrix} \\ &= \langle \theta, dX + \frac{1}{2} \Gamma(X) dX d\overline{X} \rangle_{E^*, E} \text{ in the chart.} \end{aligned}$$

This result is due to P. A. Meyer. [33], [34], [37] are in the bibliographical index at the end of the book, page 125.

In the same way, if  $X, Y$  are semimartingales  $M, N$ , they have differentials  $\overline{dX}, \overline{dY}$  according to (4.7). Therefore, one may write a SDE by writing

$$(7.6) \quad \overline{dY} = e(X, Y) \overline{dX}, Y_0 = y.$$

And Every proves that the properties are the same.

One can prove that there exists one and only one Schwartz morphism  $f$  of the type (6.5), equal to  $e$  on  $T(M)$ , commuting with the given connections. This means that the image by  $f(x, y)$  of a (vertical or) horizontal element of  $\tau_x(M)$  has the same property in  $\tau_y(N)$ . Then (7.6) is equivalent to (7.4) again. This property, due to Emery, isn't given in the book, but in a further article of Emery which has not yet appeared. (Emery asked me to point out

that the beginning of the last paragraph on page 88 of his book is trivially false. He gives the necessary corrections in his not yet appeared article.)

This book ends with a chapter on parallel transports, moving frames, lifting, and developments with respect to a connection; it's also very rich in various results. Most of them are known (it's one of the first ideas of Malliavin about the transfer principle: parallel transport can be done along a smooth curve, therefore also along a semimartingale) but all are expounded following the ideas explained before: all the weapons have been now fully prepared for such an exposition.

I apologize for such a long analysis. I think the book deserves it, and I hope it will help people to read stochastic infinitesimal calculus without tears!

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*The topology of 4-manifolds*, by Robion C. Kirby. Springer-Verlag, Berlin, New York, 1989, 106 pp., \$13.50. ISBN 3-540-51148-2

In the years between roughly 1975 and 1985, the modern theory of four-dimensional manifolds was born as workers came to recognize that there was a fundamental difference between the topological theory of these manifolds and the corresponding smooth theory. One of the most striking aspects of this difference is that there are smooth manifolds homeomorphic to 4-dimensional euclidean space which are not diffeomorphic to it, a phenomenon which happens in no other dimension.

During the 1950s and 60s, great progress was made on basic existence and classification questions for manifolds in dimensions greater than 4. Thom's theory of transversality and Smale's theory of handlebodies were used to reduce many outstanding problems to a mixture of algebraic  $K$ -theory and homotopy theory. Throughout much of this period the results applied only to smooth or PL manifolds, but in 1969, Kirby and Siebenmann were able to prove that these transversality and handlebody techniques were also valid for topological manifolds.