may lead to a complete solution of Bannai's problem to determine all $\mathscr{Q}$-polynomial DRGs of diameter > 2.

There is no doubt that the book under review will be an essential tool for the specialists in discrete mathematics. But also the general mathematician may take advantage of the ideas expressed in the book. To illustrate this we recall that in the very first line of this review we spoke of regularity, and not of symmetry of the platonic solids. Indeed, the book is devoted to graphs having well-defined regularity properties. In the group case regularity actually can be interpreted as symmetry. This is important for two reasons. Group theory provides many examples of nice graphs; and algebraic graph theory sometimes provides interesting results about groups.

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The Markoff and Lagrange spectra, by Thomas W. Cusick and Mary E. Flahive. Math. Surveys and Monographs, no. 30, Amer. Math. Soc., Providence, RI, 1989, ix + 97 pp., $42.00. ISBN 0-8218-1531-8

Diophantine approximation begins with the following theorem of Dirichlet: Let α be a real number and $Q \geq 1$ an integer, then there exist integers p and q such that $1 \leq q \leq Q$ and $|\alpha q - p| \leq (Q+1)^{-1}$. From this basic result there springs a large number of generalizations, extensions and variations. Suppose, for example, that $\|x\|$ denotes the distance from the real number x to the nearest integer. If α is irrational, then it follows immediately that there are infinitely many positive integers q which satisfy

$$q\|\alpha q\| < 1. \tag{1}$$

Naturally one may ask if the bound in (1) can be sharpened and it is a result of Hurwitz [5] (and implicit in an earlier paper of Markoff [8]) that it can be. In fact if α is irrational, there are infinitely many positive integers q such that

$$q\|\alpha q\| < 5^{-1/2} \tag{2}$$

and here the constant $5^{-1/2}$ is best possible. More precisely, if α is equivalent under the natural action of $PGL(2, \mathbf{Z})$ to a root of $x^2+x-1=0$, then $\liminf_{q\to\infty} q\|\alpha q\| = 5^{-1/2}$. Suppose, however, that α is irrational but α is not equivalent under the action of $PGL(2, \mathbf{Z})$ to a root of $x^2+x-1=0$. As was shown by Markoff, for such α there are infinitely many positive integers q which satisfy $q\|\alpha q\| < 8^{-1/2}$ and now the constant $8^{-1/2}$ is best possible. For if α is equivalent under the action of $PGL(2, \mathbf{Z})$ to a root of $x^2+2x-1=0$, then $\liminf_{q\to\infty} q\|\alpha q\| = 8^{-1/2}$. It turns out that results of this type can be continued indefinitely. For there exists a sequence of irreducible quadratic polynomials $f_1(x)$, $f_2(x), \ldots$ in $\mathbf{Z}[x]$ and a corresponding sequence $c_1 > c_2 > c_3 > \ldots$ of positive constants such that if α is irrational but not equivalent to a root of f_1, $f_2, \ldots$, or f_{N-1}, then

$$q\|\alpha q\| < c_N \tag{3}$$

holds for infinitely many integers $q \geq 1$. The first few polynomials are $f_1(x) = x^2+x-1$, $f_2(x) = x^2+2x-1$, $f_3(x) = 5x^2+11x-5$, $f_4(x) = 13x^2+29x-13$, and the first few constants are $c_1 = 5^{-1/2}$, $c_2 = 8^{-1/2}$, $c_3 = 5(221)^{-1/2}$, $c_4 = 13(1517)^{-1/2}$. Here c_N is such that if α is equivalent to a root of f_N, then $\liminf_{q\to\infty} q\|\alpha q\| = c_N$ and therefore the bound (3) is best possible.

This line of investigation can be more easily pursued by introducing the function $\alpha \to \mu(\alpha)$, which is defined for irrational real α by $\mu(a) = \liminf_{q\to\infty} q\|\alpha q\|$. Usually the set of values taken on by $\mu(\alpha)$ is called the Lagrange spectrum. However, in the book under review and therefore in the present review, the authors find it more convenient to define the *Lagrange spectrum* to be the set

$$\mathbf{L} = \{\mu(\alpha)^{-1} : \alpha \in \mathbf{R}\backslash\mathbf{Q} \quad \text{and} \quad 0 < \mu(\alpha)\}.$$

In view of our previous remarks the first few small values in $\mathbf{L}$ are $(c_1)^{-1} = 5^{1/2}$, $(c_2)^{-1} = 8^{1/2}, \ldots$ and these converge to the smallest limit point of L which is 3. The set $\mathbf{L}$ is a rather complicated closed subset of $(0, \infty)$ whose elements determine the best constants in Diophantine inequalities involving the integer multiples of irrational numbers. Not surprisingly $\mathbf{L}$ can also be defined using continued fractions.

There is another closed subset $\mathbf{M}$ of positive real numbers, called the Markoff spectrum, which is closely related to $\mathbf{L}$. The set $\mathbf{M}$ occurs naturally from considerations in the geometry of numbers. Let $f(x, y) = ax^2 + bxy + cy^2$ be an indefinite binary quadratic form with real coefficients and discriminant $d(f) = b^2 - 4ac > 0$. We define the minimum $m(f)$ of the form by

$$m(f) = \inf\{|f(x, y)| : x \in \mathbf{Z}, y \in \mathbf{Z}, \quad \text{and} \quad (x, y) \neq (0, 0)\}.$$

The set of values taken on by the expression $m(f)(d(f))^{-1/2}$ is commonly used as the definition of the Markoff spectrum. In the book under review it is more convenient and consistent with the authors' definition of the Lagrange spectrum to use the reciprocals of these numbers. Thus the *Markoff spectrum* is defined by

$$\mathbf{M} = \{(d(f))^{1/2}(m(f))^{-1} : f \text{ is an indefinite binary quadratic form with real coefficients and } 0 < m(f)\}.$$

The relationship between $\mathbf{L}$ and $\mathbf{M}$ can be seen by defining each in terms of certain continued fractions. We recall that if $\{b_0, b_1, b_2, \dots\}$ is a sequence of integers with $b_n \geq 1$ for $n \geq 1$, then

$$\begin{aligned} \beta &= [b_0, b_1, b_2, \dots] \\ &= \lim_{N \to \infty} [b_0, b_1, \dots, b_N] \\ &= \lim_{N \to \infty} b_0 + \cfrac{1}{b_1 + \cfrac{1}{\cdots \cfrac{1}{b_N}}} \end{aligned}$$

is the simple continued fraction expansion for the real number β having $\{b_0, b_1, b_2, \dots\}$ as its sequence of partial quotients. Now let $\mathbf{A}$ denote the set of doubly infinite sequences $A = \{\dots, a_{-2}, a_{-1}, a_0, a_1, a_2, \dots)$ of positive integers. If $A \in \mathbf{A}$ and $n \in \mathbf{Z}$ we define

$$\lambda_n(A) = [a_n, a_{n+1}, a_{n+2}, \dots] + [0, a_{n-1}, a_{n-2}, \dots]$$

so that $\lambda_n(A)$ is always a positive real number. For each $A \in \mathbf{A}$ we set

$$\begin{aligned} L(A) &= \limsup\{\lambda_n(A) : n \in \mathbf{Z}\}, \\ M(A) &= \sup\{\lambda_n(A) : n \in \mathbf{Z}\}. \end{aligned}$$

Then it was shown by Perron [12] that $\mathbf{L}$ and $\mathbf{M}$ occur as

$$\mathbf{L} = \{L(A) : A \in \mathbf{A} \quad \text{and} \quad L(A) < \infty\}, \tag{4}$$

$$\mathbf{M} = \{M(A) : A \in \mathbf{A} \quad \text{and} \quad M(A) < \infty\}. \tag{5}$$

It is this approach to the sets $\mathbf{L}$ and $\mathbf{M}$ which the authors employ throughout most of their book to give a reasonably complete account of current knowledge of these spectra. For example, it follows easily from (4) and (5) that $\mathbf{L} \subseteq \mathbf{M}$ and that $\mathbf{L} \cap (0, 3)$ and $\mathbf{M} \cap (0, 3)$ are equal. In 1947 M. Hall [4] proved that every real number can be written in the form

$$a + [0, b_1, b_2, \ldots] + [0, c_1, c_2, \ldots],$$

where a is an integer and the partial quotients $\{b_1, b_2, \ldots\}$ and $\{c_1, c_2, \ldots\}$ are positive integers less than or equal to 4. From this a simple argument shows that $\mathbf{L}$, and therefore $\mathbf{M}$, contains an infinite closed interval $[\gamma, \infty)$. In 1975, in a deep and difficult paper, G. A. Freiman [3] found the smallest value of γ for which this is so. At present Freiman's result requires heroic calculations so lengthy that they have understandably not been included in the present volume. The precise result is that $[\gamma, \infty) \subseteq \mathbf{L} \subseteq \mathbf{M}$, with

$$\begin{aligned} \gamma &= \frac{253, 589, 820 + (283, 748)\sqrt{462}}{491, 993, 569} \\ &= 4.52782956\cdots, \end{aligned}$$

but there exists a nonempty open interval (δ, γ) which contains no point of $\mathbf{M}$.

While our understanding of $\mathbf{L}$ and $\mathbf{M}$ is not complete, it is quite substantial. Virtually all results can be obtained from patient, sometimes painstaking, analysis with continued fractions. Happily, the authors carefully and expertly clarify the occasionally muddled history of research on the spectra. The subject matter here is admittedly rather specialized and at times one feels somewhat cut off from other mathematical theories. A refreshing exception occurs in the last chapter of the book where results on $\mathbf{L}$ and $\mathbf{M}$ are obtained using the modular group $\Gamma = SL(2, \mathbf{Z})$. Thus Hurwitz's theorem (2) is established directly from knowledge of the familiar fundamental domain for the action of Γ on the upper half-plane $H = \{z \in \mathbf{C} : \operatorname{Im}(z) > 0\}$. This approach was first found by L. R. Ford [1, 2].

There are two famous problems which are not considered in the volume being reviewed but which deserve to be mentioned. Suppose that α_1 and α_2 are both irrational real numbers. Of course it may happen that $\mu(\alpha_1) > 0$ and $\mu(\alpha_2) > 0$. On the other hand Littlewood has conjectured that $\liminf_{q\to\infty} q\|\alpha_1 q\|\|\alpha_2 q\| = 0$, and this remains a difficult open question. Next let f be a real nondegenerate indefinite quadratic form in $N \geq 3$ variables and set

$$m(f) = \inf\{|f(\vec{x})| : \vec{x} \in \mathbf{Z}^N \quad \text{and} \quad \vec{x} \neq \vec{0}\}.$$

If f is a multiple of a form with integer coefficients, then plainly $m(f) = 0$ precisely when f represents zero nontrivially. As is well known, such an f will always represent zero nontrivially when $N \geq 5$. If f is not a multiple of a form with integer coefficients, then it was a long outstanding conjecture of Oppenheim [9, 10] that $m(f) = 0$. This was recently settled by G. A. Margulis [6, 7] in the slightly stronger form (also conjectured by Oppenheim [11]) that for every $\varepsilon > 0$ there exists $\vec{x} \in \mathbf{Z}^N$ with $0 < |f(\vec{x})| < \varepsilon$.

For those unfamiliar with previous work on the spectra we have already noted that the book under review uses a definition of **L** and **M** which is the reciprocals of the numbers more commonly used to define these sets. Attention should also be called to a typographical error on page 1 (and repeated on page 85) which occurs in the definition of the function $\alpha \to \mu(\alpha)$. In summary, the reviewer found this volume to be interesting and well researched. It should become a standard reference for results on the spectra.

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Barrelled locally convex spaces, by P. Pérez Carreras and J. Bonet. North-Holland Math. Stud., vol. 131, (L. Nachbin, ed.), North-Holland, Amsterdam, New York, Oxford, Tokyo, 1987, xv +512 pp., $80.00. ISBN 0-444-70129-X

The book targeted by this review is truly a landmark, projecting barrelledness as the major force which disciplines the general theory of locally convex topological vector spaces (lctvs); it is the definitive study of strong and weak barrelledness structure theory, with a novel choice of applications (Chapters 10–12); it is a unique assimilation of half a century's scholarship, comprehensive, coherent, current, with a marvelous collection of open problems (Chapter 13). In his fanciful flight over an area he has roosted in for two decades, the reviewer finds the book the monument which most powerfully stimulates and facilitates fresh contributions (see below), contributions particularly urgent in view of the fertile open problems and certain creatively correctable mistakes (a modest price for such timeliness). A clearer bird's-eye view of strong barrelledness unfolds (see Figures 1–4).

The recent demise of the beloved patriarch, Prof. Dr. G. Köthe, recalls the historically grand German tradition in topological vector spaces (tvs). Now Spain has emerged a leader with the advent of Prof. M. Valdivia and his prolific followers, including the authors P. Pérez Carreras and J. Bonet. Unfortunately, some of their important results appear in Spanish publications not widely available. Fortunately, the book under review redresses this situation beautifully, transporting *to* the New World a cargo richer than the plunder of conquistadors. As in a Goya painting, the book's subject is robust and beguilingly composed. Although some knowledge of tvs's is requisite, to most in the area the book is necessary and sufficient. Chapters 10, 11, and 12, respectively, make