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Distance-Regular Graphs, by A. E. Brouwer, A. M. Cohen, and A. Neumaier. Springer-Verlag, New York, 1989, 485 pp., $84.50. ISBN 3-540-50619-5

The title of the present book refers to a class of graphs whose regularity properties go back to the platonic solids of antiquity. Examples of such graphs are provided by the vertices and the edges of the cube, or of the icosahedron. On the other hand, the graphs have deep connections to many topics of the present-day theory and applications of groups, geometries, codes, and designs. Thus the book represents a large part of discrete mathematics. The geometry of the graphs is phrased in terms of distances; it has a direct translation into algebra. Many mathematical disciplines, ranging from functional analysis to computation, contribute to the understanding of our graphs. Now let us first give some definitions. The cube graph serves as a trivial example; we will meet other examples later.

Let $\Gamma = (V, E)$ denote an undirected, connected graph, with vertex set V of size v and edge set E of size e. The distance $d(\gamma, \delta)$ between any $\gamma, \delta \in V$ is measured by the minimum number of steps in Γ to get from γ to δ. Thus the pairs of the vertices of the cube graph are at distance 0, 1, 2, or 3. The regularity properties of Γ are expresssed in terms of distances as follows. Given $\gamma \in V$, the sphere $\Gamma_j(\gamma)$ consists of the vertices at distance j from γ. The graph Γ is called *regular* if the valency $k := |\Gamma_1(\gamma)|$ is a constant for $\gamma \in V$. The graph Γ is called a *distance-regular graph* (DRG for short) if, for each distance j, each of the cardinalities

$$a_j := |\Gamma_j(\gamma) \cap \Gamma_1(\delta)|, \qquad b_j := |\Gamma_{j+1}(\gamma) \cap \Gamma_1(\delta)|,$$
$$c_j := |\Gamma_{j-1}(\gamma) \cap \Gamma_1(\delta)|$$

is a constant for $\gamma, \delta \in V$ at distance j. Thus the cube graph has $(b_0, b_1, b_2; c_1, c_2, c_3) = (3, 2, 1; 1, 2, 3)$. The relevant parameters are the *intersection numbers* in the array $(b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d)$, where d is the diameter of the graph. Indeed,

$$b_0 = k, \qquad c_0 = 0 = b_d, \qquad c_1 = 1, \qquad k = a_j + b_j + c_j \quad \text{for } j = 0, 1, \dots, d.$$

We translate this definition into algebra. The distance-i-matrix A_i of size $v \times v$ is defined by its entries $A_i(\gamma, \delta) = 1$ if $d(\gamma, \delta) = i$ in Γ, and $A_i(\gamma, \delta) = 0$ otherwise. Then

$$A_0 = I, \qquad A_0 + A_1 + A_2 \cdots + A_d = J,$$

where I and J denote the unit matrix and the all-one matrix of size $v \times v$. In terms of A_i the *definition of DRG* reads, for $i = 0, 1, \dots, d$,

$$A_1 A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1}.$$

Notice that the matrices A_i can be written as polynomials of degree i in $A := A_1$. In addition we find

$$A_j A_i = A_i A_j = \sum_{l=0}^{d} P_{ij}^l A_l$$

for certain $P_{ij}^l \in \mathbf{N}$, that is, our DRG with its distance relation forms a (symmetric) *association scheme* with d classes. This concept orginates from the design of experiments in statistics (Bose[1]

[1] All references are to the bibliography in the text under review.

c.s. [111, 109]) and, independently, from finite permutation groups (Wielandt [782]). It was applied in coding theory by Delsarte [251], and further developed by him, Higman [386], and Bannai-Ito [33]. The concept and the theory of DRG were introduced by Biggs [71].

Important classes of DRG are the following. The *Hamming graph* $H(d, n)$ has as its vertices the words of length d over an alphabet of size n, and the (Hamming) distance between two words is the number of the coordinates in which they differ. The intersection array of the Hamming graph is $(dm, (d-1)m, \ldots, m; 1, 2, \ldots, d)$ for $m = n - 1$. For $n = 2$ this is the d-cube, and it subsets are the binary codes of length d.

The *Johnson graph* $J(n, d)$ consists of the d-subsets of an n-set, and any two d-subsets are adjacent if they intersect in $d - 1$ elements. $J(n, d)$ is a DRG and has $v = \binom{n}{d}$, $k = d(n-d)$. It is the graph underlying designs with constant block sizes.

The heart of the book, Chapters 4 through 9, gives an extensive treatment of the theory of DRG. One of the most spectacular results is contained in Theorem 6.1.1. It states that the Hamming graphs $H(d, n)$, for $n \neq 4$, and the Johnson graphs $J(n, d)$, for $(n, d) \neq (8, 2)$, are characterized by their intersection arrays. This problem has puzzled scientists from various backgrounds for a number of years. The exceptions come from statistical schemes which share the parameters, yet are nonisomorphic with $H(2, 4)$ (Shrikhande), with $H(n, 4)$ (Doob), and with $J(8, 2)$ (Chang), respectively. The characterization was finally settled by Neumaier [561] and Terwilliger [719]. We give an indication of the methods used in their proofs. Both authors find an appropriate idempotent matrix $E = \sum_{i=0}^{d} u_i A_i$ in the algebra spanned by the distance-i-matrices A_i of a DRG Γ. Viewing the symmetric matrix E as the Gram matrix of the inner products $(\cdot, \cdot)$ of v vectors $\overline{\gamma}_1, \ldots, \overline{\gamma}_v$ in a Euclidean space of dimension $f := \operatorname{rank} E$, one obtains a geometric representation of the graph Γ. Assuming we are in the case $u_i = a - bi$, we calculate the inner products of the differences of the vectors in terms of the distances of the corresponding vertices in Γ:

$$(\overline{\gamma}_p - \overline{\gamma}_q, \overline{\gamma}_r - \overline{\gamma}_s) = b(d(\gamma_p, \gamma_r) + d(\gamma_q, \gamma_s) - d(\gamma_p, \gamma_s) - d(\gamma_q, \gamma_r)).$$

This leads to an even integral lattice in Euclidean f-space:

$$\left\langle \frac{1}{\sqrt{b}}(\overline{\gamma}_p - \overline{\gamma}_q) : d(\gamma_p, \gamma_q) = 1 \right\rangle \mathbb{Z}.$$

In the case that this lattice spans the space, the root system techniques introduced by Cameron c.s. [180] can be applied. The lattice then must be a direct sum of root lattices of the well-known types A, D, and E. Eventually, this leads to the desired result, cf. Theorem 4.4.10.

After this introduction in the subject and the methods, we will briefly sketch the content of the book by going through the various chapters. The first three are preliminary. Together with the Appendix they collect the material for a separate book on graph theory and combinatorics. DRGs are treated in Chapters 4 through 9, and in the tables of Chapter 14. The remaining Chapters 10 through 13 study graphs occurring in relation to Coxeter systems, Chevalley groups, codes, classical geometries, and sporadic groups. Most valuable is the bibliography which contains 800 references. Thus this Ergebnisse book of almost 500 printed pages really presents a survey of discrete mathematics as it was developed in the last 25 years.

The book is written for the researcher. It is not a textbook. For the beginner in the field the tract by Biggs [71] of 1974 still maintains its value as an introduction. It is difficult to compare the present book with the work by Bannai-Ito, since up to now their "Algebraic Combinatorics" has only appeared in Part I, cf. [33]. The starting points of the two books are different. Chapter 1 of [33] deals with representations of finite groups; the notions of distance-regular and distance-transitive graphs are introduced in the same subsection. In the book under review the preliminary chapters are combinatorial, and the group case of distance-transitive graphs appears in Chapter 7, halfway through the book. In [33] algebraic combinatorics is described as "a character theoretic study of combinatorial objects" or as "a group theory without groups." The book under review has the same philosophy, but is more independent in its approach; it studies the group case of distance-transitive graphs of Lie type in the context of Tits systems (Chapter 10). It is interesting to observe the interaction between the books. The core of [33] is Leonard's result [487] that DRGs of $\mathscr{Q}$-polynomial type satisfy restricted parameter conditions, and have integral eigenvalues if the diameter is large. The present book performs a fine job in reworking Leonard's theory in Chapter 8. Meanwhile new bounds for the diameter have become available (Ivanov [430], Godsil [325], Terwilliger [715]), which are treated in Chapter 5. Thus, as the authors expect, these common efforts may lead to a complete solution of Bannai's problem to determine all $\mathscr{Q}$-polynomial DRGs of diameter > 2.

There is no doubt that the book under review will be an essential tool for the specialists in discrete mathematics. But also the general mathematician may take advantage of the ideas expressed in the book. To illustrate this we recall that in the very first line of this review we spoke of regularity, and not of symmetry of the platonic solids. Indeed, the book is devoted to graphs having well-defined regularity properties. In the group case regularity actually can be interpreted as symmetry. This is important for two reasons. Group theory provides many examples of nice graphs; and algebraic graph theory sometimes provides interesting results about groups.

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The Markoff and Lagrange spectra, by Thomas W. Cusick and Mary E. Flahive. Math. Surveys and Monographs, no. 30, Amer. Math. Soc., Providence, RI, 1989, ix + 97 pp., $42.00. ISBN 0-8218-1531-8

Diophantine approximation begins with the following theorem of Dirichlet: Let α be a real number and $Q \geq 1$ an integer, then there exist integers p and q such that $1 \leq q \leq Q$ and $|\alpha q - p| \leq (Q+1)^{-1}$. From this basic result there springs a large number of generalizations, extensions and variations. Suppose, for example, that $\|x\|$ denotes the distance from the real number x to the nearest integer. If α is irrational, then it follows immediately that there are infinitely many positive integers q which satisfy

$$q\|\alpha q\| < 1. \tag{1}$$

Naturally one may ask if the bound in (1) can be sharpened and it is a result of Hurwitz [5] (and implicit in an earlier paper of Markoff [8]) that it can be. In fact if α is irrational, there are infinitely many positive integers q such that

$$q\|\alpha q\| < 5^{-1/2} \tag{2}$$