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CLASSIFICATION OF SIMPLE LIE ALGEBRAS OVER ALGEBRAICALLY CLOSED FIELDS OF PRIME CHARACTERISTIC

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We announce here a result which completes the classification of the finite-dimensional simple Lie algebras over an algebraically closed field F of characteristic $p > 7$, showing that these algebras are all of classical or Cartan type. This verifies the *Generalized Kostrikin-Šafarevič Conjecture* [Kos-70, Kac-74]. Many authors have contributed to the solution of this problem. For a discussion of work before 1986 see [Wil-87]. More recent work is cited in §1.

Let F be an algebraically closed field of characteristic $p > 7$ and L be a finite-dimensional semisimple Lie algebra over F. We identify L with $\operatorname{ad} L$, the subalgebra of inner derivations of L and write $\overline{L}$ for the restricted subalgebra of $\operatorname{Der} L$ (the algebra of derivations of L) generated by L. A *torus* T in $\overline{L}$ is a restricted subalgebra such that for every element $x \in T$, $\operatorname{ad} x$ is a semisimple linear transformation on L. Let $\mathcal{T}(L)$ denote the set of tori contained in $\overline{L}$. The (*absolute*) *toral rank* of L is

$$\mathrm{TR}(L) = \max\{\dim T | T \in \mathcal{T}(L)\}.$$

(The absolute toral rank may also be defined for nonsemisimple algebras [St-89a] using the theory of p-envelopes and this extension is necessary for the proofs of the results presented here.) We say a torus $T \subseteq \overline{L}$ is of *maximal rank* in $\overline{L}$ if $\dim T = \mathrm{TR}(L)$. If $T \in \mathcal{T}(L)$ we have the root space decomposition of L with respect to T,

$$L = \sum_{\alpha \in T^*} L_\alpha$$

where

$$L_\alpha = \{x \in L | [t, x] = \alpha(t)x \quad \text{for all } t \in T\}.$$

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We then write

$$L^{(\alpha)} = \sum_{i=0}^{p-1} L_{i\alpha}$$

and say that a root α is *solvable* if $L^{(\alpha)}$ is solvable, *classical* if $L^{(\alpha)}/\operatorname{solv}(L^{(\alpha)})$ is isomorphic to $\mathrm{sl}(2)$ and *nonclassical* otherwise. (Here $\operatorname{solv}(A)$ denotes the solvable radical of the Lie algebra A.) We also write

$$L^{(\alpha,\beta)} = \sum_{i,j=0}^{p-1} L_{i\alpha+j\beta}.$$

1. General case

The finite-dimensional restricted simple Lie algebras over F were classified by Block and Wilson in [BW-88]. Strade [St-89b] extended, by use of the theory of p-envelopes, many of the results on rank two semisimple algebras proved in [BW-88] to results which can be used in the classification of not necessarily restricted algebras. Using this Benkart, Osborn, and Strade [BOSt-pre, St-pre-1] derive further results necessary for work described below.

We say that L satisfies condition $(*)$ if whenever $L^{(\alpha,\beta)}$ is nonsolvable then some root $i\alpha + j\beta$ is nonsolvable. The classification problem then divides naturally into four cases, depending on the types of roots occurring and on whether or not condition $(*)$ is satisfied. These cases are:

Case A. Every root with respect to every torus of maximal rank is solvable. We announce here the classification of these algebras. This result is discussed in §2; the proof will appear in [StW-pre].

Case B. Every root with respect to some torus T of maximal rank is solvable or classical and some root with respect to T is classical. By consideration of the representations of one of the algebras $L(G, 0, f)$ of Block [Blo-58] the first author showed that $(*)$ holds [St-pre-3, St-pre-4]. Benkart [Ben-90] and Strade [Str-pre-2] have proved independently that an algebra satisfying $(*)$ and Case B must be classical.

Cases C *and* D. There is a nonclassical root with respect to some torus of maximal rank. Cases C and D are distinguished by whether $(*)$ holds for certain tori or not. The analysis of Cases C and D in [St-pre-5] derives detailed information about the Cartan-Weisfeiler filtration [Wei-68] and then uses results on the associated graded algebra due to Weisfeiler [Wei-78]. In each case it is shown that the algebra must be of Cartan type by application of an appropriate form of the Recognition Theorem for Cartan type algebras. In Case C the form needed is that stated as Theorem 1.2.2 of [BW-88] (see [Kac-71, Kac-74, Wil-76]; for a generalization see [BG-89]). In Case D it is Theorem 1 in [Wil-76].

Since Cases A–D are exhaustive, the classification of the algebras satisfying the hypotheses of Case A completes the general classification.

2. Algebras with solvable roots

We now describe the analysis of Case A. In other cases of the classification problem the primary objects of study are the semisimple algebras of toral ranks 1 and 2. However, if all roots are solvable then a rank 1 algebra cannot be semisimple. This suggests that for Case A one may need to study semisimple algebras of toral ranks 2 and 3.

The semisimple algebras of toral rank 2 satisfying the conditions of Case A have been determined [Proposition 2.2, St-pre-4]. There is only one such algebra, namely $H(2:\mathbf{1}:\Phi(\tau))^{(1)}$ where $\Phi(\tau)$ has the property

$$\Phi(\tau)(dx_1 \wedge dx_2) = (x_1+1)^{p-1}(x_2+1)^{p-1}\,dx_1 \wedge dx_2 .$$

The nonsimple semisimple algebras of toral rank 3 satisfying the conditions of Case A have also been determined [Theorem 2.7, St-pre-4]. The next two propositions determine the simple algebras of toral rank 3 which satisfy the conditions of Case A.

Proposition 1. *Let L be a finite-dimensional simple Lie algebra over F which satisfies the conditions of Case* A *and is of toral rank* 3. *Then L is of Cartan type.*

To prove this result one notices that, since $H(2:\mathbf{1}:\Phi(\tau))^{(1)}$ contains a one-dimensional maximal torus, $\overline{L}$ must contain a two-dimensional maximal torus. Then the techniques of Chapters 5–9 of [BW-88] may be applied to L to obtain the desired result.

The following proposition, which is proved by reduction to the cases $X(m:\mathbf{1}:\Phi)$, $X = W, S, H, K$, by use of results of Demuskin [Dem-70, Dem-72] and by explicit computation, shows which algebras of Cartan type can occur.

Proposition 2. *Let L be a finite-dimensional simple Lie algebra of Cartan type over F which satisfies the conditions of Case* A *and* *is of toral rank* 3. *Then either* $L \cong H(2:(2,1):\Phi(\tau))^{(1)}$ *or* $L \cong S(3:\mathbf{1}:\Psi)^{(1)}$ *where*

$$\Psi(dx_1 \wedge dx_2 \wedge dx_3) \\ = (x_1+1)^{p-1}(x_2+1)^{p-1}(x_3+1)^{p-1}\, dx_1 \wedge dx_2 \wedge dx_3 .$$

In either case, if α is any root with respect to a torus of maximal rank then $L^{(\alpha)}$ is abelian and every element of L_α is semisimple.

We now define two types of Lie algebras by giving bases and multiplication tables. The first construction is due to Block [Blo-58], the second is similar to Kaplansky's construction of generalized Witt algebras [Kap-54].

First let G be an elementary abelian group of order p^n and $f(\cdot,\cdot)$ be an F-valued, biadditive, alternate form on G. Assume that f is nondegenerate (i.e., $\alpha \in G$, $f(\alpha, G) = (0)$ implies $\alpha = 0$). Let V be a vector space with basis $\{v_\alpha | \alpha \in G\}$. Define a bilinear product on V by

$$[v_\alpha, v_\beta] = f(\alpha, \beta) v_{\alpha+\beta} .$$

Then V is a Lie algebra, Fv_0 is its center and $L(G, 0, f) = V/Fv_0$ is a simple Lie algebra. It is easily seen that $\mathrm{TR}(L(G, 0, f)) = n$ and that $L(G, 0, f)$ satisfies the conditions of Case A. If $n = 2$ this algebra is isomorphic to $H(2:\mathbf{1}:\Phi(\tau))^{(1)}$.

Second let M be an m-dimensional vector space and let G be an additive subgroup of M^* of order p^n where $m \geq 3$ and $\bigcap_{\mu\in G} \ker(\mu) = (0)$. For $\alpha \in G$ let $M_\alpha = \{\alpha\} \times \ker(\alpha)$ be a vector space isomorphic to $\ker(\alpha)$. Give $V(M, G) = \sum_{\alpha\in G, \alpha\neq 0} M_\alpha$ the structure of an algebra by defining

$$[(\alpha, s), (\beta, t)] = (\alpha+\beta, \alpha(t)s - \beta(s)t) .$$

Then $V(M, G)$ is a simple Lie algebra. It is easily seen that $\mathrm{TR}(V(M, G)) = n$ and that $V(M, G)$ satisfies the conditions of Case A. If $m = n = 3$ this algebra is isomorphic to $S(3:\mathbf{1}:\Psi)^{(1)}$.

Using the result of Proposition 2 and the determinations of the rank 2 semisimple and nonsimple rank 3 semisimple algebras cited above, one can prove the following proposition.

Proposition 3. *Let* L *be a finite-dimensional simple Lie algebra over* F *which satisfies the conditions of Case* A. *Then* L *is isomorphic to some algebra* $L(G, 0, f)$ *or* $V(M, G)$.

The classification of the algebras satisfying the conditions of Case A is then completed by the following result.

Proposition 4. *Any algebra $L(G, 0, f)$ is isomorphic to an algebra of type H. Any algebra $V(M, G)$ is isomorphic to an algebra of type S.*

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