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HYPERGEOMETRIC FUNCTIONS ON COMPLEX MATRIX SPACE

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0. Introduction

In [6] we presented the general foundations for a theory of hypergeometric functions of matrix argument over real division algebras. In this note, we further develop the fine structure of these functions over the complex field, including series expansions, integral representations, asymptotics, differential equations, addition formulas, multiplication formulas, summation theorems, transformation properties, etc. Especially important in this paper are the *operator-valued hypergeometric functions*, required for (nonspherical) expansions such as addition formulas by the noncommutativity of matrix multiplication. These functions generalize the operator-valued Bessel functions studied in [5].

Hypergeometric functions of matrix argument arise naturally in applications ranging from multivariate statistics, quantum physics, and molecular chemistry, to harmonic analysis, group representations, and number theory. (See the references in [6].) These diverse applications amplify the need to develop the fine structure systematically and to the greatest extent possible.

We briefly review the definition of hypergeometric functions of matrix argument from [6]. Let $\mathbf{F}$ be the real field, the complex field, or the quaternions. Denote by S the space of all $n \times n$ Hermitian matrices $s = s^*$ over $\mathbf{F}$, on which the group $G = GL(n, \mathbf{F})$ of invertible $n \times n$ matrices g over $\mathbf{F}$ acts by $s \mapsto g^* s g$. Then $K = \{k \in G : k^* k = 1\}$ is the isotropy subgroup of the identity matrix 1, the open cone P in S of positive-definite $n \times n$ matrices is the orbit under G of 1, and P can be identified with the symmetric space $K \backslash G$. A functions f on S is *K-invariant* if

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$f(k^{-1}sk) = f(s)$ for all $s \in S$ and $k \in K$.

An n-tuple $m = (m_1, \ldots, m_n)$ of nonnegative integers such that $m_1 \geq \cdots \geq m_n$ is called a *partition*. Set $|m| = m_1 + \cdots + m_n$. As a G-module the algebra of polynomial functions on S is multiplicity free, the irreducible components are indexed by the partitions m, and in the mth irreducible component is a K-invariant polynomial Z_m on S, homogeneous of degree $|m|$ and unique up to scalar multiples, called a *zonal polynomial*. The zonal polynomials are normalized by the condition $(\operatorname{tr} s)^d = \sum_{|m|=d} Z_m(s)$. In the terminology of harmonic analysis, the zonal polynomials are *polynomial spherical functions* for the symmetric space $K\backslash G$.

Let $a_1, \ldots, a_p$ and $b_1, \ldots, b_q$ be complex parameters. From [6, (6.1.1)], the hypergeometric function ${}_pF_q$ of matrix argument is a K-invariant function defined on S by

$$(1) \quad {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; s) = \sum_m \frac{[a_1]_m \cdots [a_p]_m}{[b_1]_m \cdots [b_q]_m} \frac{Z_m(s)}{|m|!}$$

where $[a]_m = \prod_{i=1}^n (a - \frac{1}{2}(i-1)\nu)_{m_i}$ is the *generalized truncated factorial* for the matrix space [6; (5.7.2)], $(a)_k$ denotes the ordinary truncated factorial, and $\nu = \dim_{\mathbf{R}} \mathbf{F}$. In analogy to the classical case, ${}_0F_0(s) = e^{\operatorname{tr} s}$ and ${}_1F_0(a; -; s) = \Delta(1-s)^{-a}$.

Hypergeometric functions have also been defined along the above lines on *domains of positivity*, a context which, in addition to the matrix spaces, also includes the Minkowski spaces (i.e., $\mathbf{R}^{n+1}$ with signature $(n, 1)$) and a certain exceptional 3×3 matrix space over the Cayley algebra [4, 10].

For the remainder of this note we set $\mathbf{F} = \mathbf{C}$, in which case $G = GL(n, \mathbf{C})$ is the complex $n \times n$ *general linear group* and $K = U(n)$ the *unitary group.*[1] Since a Hermitian matrix s can be diagonalized by an element of K, we can view the hypergeometric function (1) as a symmetric function of the eigenvalues $s_1, \ldots, s_n$ of the matrix s. Also, since the full matrix space $\mathbf{C}^{n \times n}$ is the complexification of the real space S, by analytic continuation we can view (1) as defining the hypergeometric function on $\mathbf{C}^{n \times n}$.

[1] Only for the case of complex matrix space, among all the domains of positivity, does the group G have complex structure relative to which the maximal compact subgroup K is a real form of G. This property over the complex field is the crucial structure upon which the detailed results of this paper ultimately rest.

1. Weyl's character formula

The partitions m that index the zonal polynomials also parametrize the irreducible finite-dimensional (complex) polynomial representations $\lambda = \lambda_m$ of G. Set $\chi_m(s) = \mathrm{tr}(\lambda_m(s))$. Then, over the complex field, the zonal polynomials are the normalized characters

$$Z_m(s) = \omega_m \chi_m(s) \tag{2}$$

of representations of G. By Weyl's *character formula* [12]

$$\chi_m(s) = \frac{\det(s_i^{m_j+n-j})}{V(s)} \tag{3}$$

where $V(s) = V(s_1, \ldots, s_n) = \prod_{1\le i<j\le n}(s_i - s_j)$ is the Vandermonde determinant. The constant ω_m is available from the literature [9, 10], but our methods provide a direct proof that

$$\omega_m = \frac{|m|!d_m}{[n]_m} \tag{4}$$

where d_m is the degree of λ_m. Formulas (2)–(4), together with a generalization of (1) involving *two* matrix arguments, lead to the following crucial result.

2. Explicit evaluation of the hypergeometric functions

Let ${}_p\mathscr{F}_q$ denote the classical hypergeometric function (i.e., the case $n = 1$). Then

(5)
$$ {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; s) = \frac{\det(s_i^{n-j}\,{}_p\mathscr{F}_q(a_1 - j + 1, \ldots, a_p - j + 1; b_1 - j + 1, \ldots, b_q - j + 1; s_i))}{V(s)}$$

Formula (5) allows one to transfer known properties of the classical hypergeometric functions to the matrix argument counterparts. Examples include precise asymptotic information and systems of differential equations, of which the following are representative samples.

3. Asymptotic formula as $s_i \to \infty$ for $i = 1, \ldots, n$

Suppose $p \le q$ and the series (1) does not terminate. Let $\beta = q + 1 - p$, $a = \sum_{i=1}^p a_i$, and $b = \sum_{i=1}^q b_i$. Then

$$(6)\quad \prod_{i<j}\left(\frac{s_i^{1/\beta}-s_j^{1/\beta}}{s_i-s_j}\right)^{-1} {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;s)$$
$$\sim \frac{(2\pi^n)^{(1-\beta)n/2}}{\beta^{n/2}}\frac{\prod_{i=1}^q\Gamma_n(b_i)}{\prod_{i=1}^p\Gamma_n(a_i)}(\det s)^{[(\beta-1)(n-1)+a-b]/\beta}$$
$$\times \exp\left(\beta\sum_{i=1}^n s_i^{1/\beta}\right).$$

Moreover, the error term is $O(\sum_{i=1}^n s_i^{-1/\beta})$.

In (6), and throughout, $\Gamma_n(a)=\pi^{n(n-1)/2}\prod_{i=1}^n\Gamma(a+m_i-(i-1))$ is the *(generalized) gamma function for the cone* P [6; (5.6.1)].

4. Differential equations

For any subset A of $\{1,\ldots,n\}$, define the differential operators

$$(7)\quad \mathfrak{D}_A=\left[\prod_{k\in A}\partial_k\prod_{i=1}^q(\partial_k+b_i-n)\right]\left[\prod_{k\notin A}s_k\prod_{i=1}^p(\partial_k+a_i-n+1)\right]$$

where $\partial_k=s_k\partial/\partial s_k$, and set

$$(8)\qquad \mathfrak{L}_j=V(s)^{-1}\left[\sum_{|A|=j}\mathfrak{D}_A-\binom{n}{j}\mathfrak{D}_\phi\right]V(s)$$

for $j=1,\ldots,n$ (where ϕ denotes the empty set). Then—subject to conditions of symmetry, analyticity, and normalization at the origin—the hypergeometric function (1) is the unique solution to the system $\mathfrak{L}_jF=0$ of n partial differential equations.

Note that the formulas in the previous two paragraphs reduce to well known classical results when $n=1$. We remark that formula (5) also yields new information about classical hypergeometric functions through use of the matrix-argument versions [7, 11].

5. Operator-valued hypergeometric functions on $\mathbf{C}^{n\times n}$

Let Δ denote the determinant and d_*r be G-invariant measure on P. Fix a partition m and the associated representation $\lambda=\lambda_m$ of G acting in a Hilbert space $\mathscr{V}$, let $\mathscr{L}(\mathscr{V})$ be the space of linear transformations on $\mathscr{V}$, and define a family of hypergeometric functions on $\mathbf{C}^{n\times n}$ with values in $\mathscr{L}(\mathscr{V})$, starting with

$$(9)\qquad {}_0F_0(\lambda|z)=e^{\operatorname{tr} z}I$$

where I is the identity on $\mathscr{V}$. For larger values of p and q, we employ inductively the definitions

(10)
$$\Delta(z)^{-a_{p+1}}\lambda(z)^{-1}{}_{p+1}F_q(\lambda|a_1,\ldots,a_{p+1};b_1,\ldots,b_q;z^{-1})$$
$$=\frac{1}{\Gamma_n(a_{p+1})}\int_P e^{-\operatorname{tr}zr}\Delta(r)^{a_{p+1}}\lambda(r){}_pF_q(\lambda|a_1,\ldots,a_p;b_1,\ldots,b_q;r)d_*r$$

and

(11)
$$\Delta(z)^{-b_{q+1}}\lambda(z)^{-1}{}_pF_q(\lambda|a_1,\ldots,a_p;b_1,\ldots,b_q;z^{-1})$$
$$=\frac{1}{\Gamma_n(b_{q+1})}\int_P e^{-\operatorname{tr}zr}\Delta(r)^{b_{q+1}}\lambda(r){}_pF_{q+1}(\lambda|a_1,\ldots,a_p;b_1,\ldots,b_{q+1};r)d_*r.$$

We call ${}_pF_q(\lambda|a_1,\ldots,a_p;b_1,\ldots,b_q;z)$ the operator-valued hypergeometric function on $\mathbf{C}^{n\times n}$ of weight λ. Note the covariance property

$$\begin{aligned}&{}_pF_q(\lambda|a_1,\ldots,a_p;b_1,\ldots,b_q;gzg^{-1})\\&=\lambda(g){}_pF_q(\lambda|a_1,\ldots,a_p;b_1,\ldots,b_q;z)\lambda(g)^{-1}\end{aligned}\tag{12}$$

for $g\in G$.

We can establish analogues for operator-valued hypergeometric functions of matrix argument of familiar classical formulas. For example, *the operator-valued binomial theorem* ${}_1F_0(\lambda|a;-;z)=[a]_m\Delta(1-z)^{-a}\lambda(1-z)^{-1}$ holds for $\operatorname{Re}z>1$. The *operator-valued Bessel function,* defined in [5; (4.1)] by an integral over K, is given by $J_\lambda(2r^{1/2})=\pi^{n(n-1)/2}\lambda(r^{1/2}){}_0F_1(\lambda|-;n;-r)$ for $r\in P$. *The operator-valued confluent hypergeometric function* has an Euler integral representation

(13)
$${}_1F_1(\lambda|a;b;z)$$
$$=\frac{\Gamma_n(b)}{\Gamma_n(a)\Gamma_n(b-a)}\int_{0<r<1}e^{\operatorname{tr}zr}\Delta(r)^a\Delta(1-r)^{b-a-n}\lambda(r)d_*r$$

for all $z\in\mathbf{C}^{n\times n}$, valid for $\operatorname{Re}a>n-m_n-1$ and $\operatorname{Re}(b-a)>n-1$; and *the operator-valued Gaussian hypergeometric function* has an Euler integral representation

(14)

$${}_2F_1(\lambda|a,b;c;z)$$
$$=\frac{\Gamma_n(c)}{\Gamma_n(b)\Gamma_n(c-b)}[a]_m\int_{0<r<1}\Delta(r)^b\Delta(1-r)^{c-b-n}\Delta(1-zr)^{-a}\lambda(r)\lambda(1-zr)^{-1}d_*r$$

for $\operatorname{Re} z < 1$, which holds when $\operatorname{Re} b > n - m_n - 1$ and $\operatorname{Re}(c-b) > n-1$.

6. Operator-valued differentiation

By applying differential operators to the scalar-valued hypergeometric functions, we obtain the analytic continuation of the operator-valued hypergeometric functions. Specifically, if $\partial/\partial z = (\partial/\partial z_{ij})$, then

$$(15)\quad \begin{aligned} &{}_pF_q(\lambda | a_1, \dots, a_p; b_1, \dots, b_q; z) \\ &\quad = \lambda(\partial/\partial z)\, {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z). \end{aligned}$$

7. Addition formulas

For z, w, and $z+w$ in the domain of the hypergeometric function,

$$(16)\quad \begin{aligned} &{}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z+w) \\ &\quad = \sum_m \frac{\omega_m}{|m|!} \operatorname{tr}(\lambda_m(z)\, {}_pF_q(\lambda_m | a_1, \dots, a_p; b_1, \dots, b_q; w)). \end{aligned}$$

The analogous *multiplication formula*

$$(17)\quad \begin{aligned} &{}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; zw) \\ &\quad = \sum_m \frac{\omega_m}{|m|!} \frac{[a_1]_m \cdots [a_p]_m}{[b_1]_m \cdots [b_q]_m} \operatorname{tr}(\lambda_m(z)\lambda_m(w)) \end{aligned}$$

is rather trivial.

8. Concluding remarks

The above results—together with generalizations of other aspects of the classical theory, such as summation theorems (e.g., Gauss' and Saalschutz' formulas, Kummer's and Thomae's transformations, etc.), orthogonal polynomials, Mellin-Barnes representations, differentiation properties, and contiguous relations—will be treated in full detail in [8]. A second paper, jointly with H. Ding, will develop a number of the new constructs—e.g., the operator-valued hypergeometric functions and differential operators—for all domains of positivity. In other related work, [3] applies the operator-valued Bessel functions on domains of positivity to representation theory of the automorphism groups of Hermitian symmetric spaces of tube type, and [1] and [2] initiate the study of operator-valued hypergeometric functions on Siegel domains of type II.

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