RESEARCH ANNOUNCEMENTS

DISTRIBUTION RIGIDITY FOR UNIPOTENT ACTIONS ON HOMOGENEOUS SPACES

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In this paper we study distributions of orbits of unipotent actions on homogeneous spaces.

More specifically, let G be a real Lie group (all groups in this paper are assumed to be second countable) with the Lie algebra $\mathfrak{G}$, Γ a discrete subgroup of G and $\pi: G \to \Gamma\backslash G$ the projection $\pi(g) = \Gamma g$, $g \in G$. The group G acts by right translations on $\Gamma\backslash G$, $(x, g) \to xg$, $x \in \Gamma\backslash G$, $g \in G$. We say that Γ is a lattice in G if there is a finite G-invariant measure on $\Gamma\backslash G$.

Let U be a subgroup of G and $x \in \Gamma\backslash G$. We say that the closure $\overline{xU}$ of the orbit xU in $\Gamma\backslash G$ is *homogeneous* if there is a closed subgroup $H \subset G$ such that $U \subset H$, $xHx^{-1} \cap \Gamma$ is a lattice in xHx^{-1}, $x \in \pi^{-1}\{x\}$, and $\overline{xU} = xH$. If these conditions are satisfied, we shall say that $\overline{xU}$ is homogeneous with respect to H.

Definition 1. A subgroup $U \subset G$ is called *topologically* rigid if given any lattice $\Gamma \subset G$ and any $x \in \Gamma\backslash G$ the closure of the orbit xU in $\Gamma\backslash G$ is homogeneous.

A subgroup $U \subset G$ is called unipotent if for each $u \in U$ the map $\mathrm{Ad}_u: \mathfrak{G} \to \mathfrak{G}$ is a unipotent automorphism of $\mathfrak{G}$.

Raghunathan's Topological Conjecture. *Every unipotent subgroup of a connected Lie group G is topologically rigid.*

Various versions of this conjecture were stated in [D1] and [M]. It was shown in [F1] and [P] that if **G** is nilpotent then every subgroup of **G** is topologically rigid. As to semisimple **G** it was shown in [F2] and [DS] that for $\mathbf{G} = SL(2, R)$ the conjecture is true. Also it was shown in [DM] that certain unipotent subgroups of $SL(3, R)$ are topologically rigid. To the best of my knowledge these are the only cases of semisimple Lie groups for which the conjecture has been settled.

In this paper we announce the following

Theorem A. *Every unipotent subgroup of a connected Lie group* **G** *is topologically rigid.*

Our method of the proof of Theorem A is totally different from that used in [DM] for certain unipotent subgroups of $SL(3, R)$. We show that in order to prove Theorem A it suffices to prove it for one-parameter unipotent subgroups **U** of **G**. For such **U** we prove, in fact, a far stronger Theorem B. To state it we need to introduce some notations.

Definition 2. Let Γ be a discrete subgroup of **G** and $\mathbf{U} = \{\mathbf{u}(t) = \exp t\mathbf{u}: t \in R\}$, $\mathbf{u} \in \mathfrak{G}$ a one-parameter subgroup of **G**. A point $x \in \Gamma\backslash\mathbf{G}$ is called *generic* for **U** if there exists a closed subgroup $\mathbf{H} \subset \mathbf{G}$ such that $\overline{x\mathbf{U}} = x\mathbf{H}$ is homogeneous with respect to **H** and

$$(*) \qquad \lim_{t\to\infty}\left(\int_0^t f(x\mathbf{u}(s))\, ds/t\right) = \int_{\Gamma\backslash\mathbf{G}} f\, d\nu_{\mathbf{H}}$$

for all bounded continuous functions f on $\Gamma\backslash\mathbf{G}$, where $\nu_{\mathbf{H}}$ denotes the **H**-invariant Borel probability measure on $\Gamma\backslash\mathbf{G}$ supported on $x\mathbf{H}$.

Similarly, for $\mathbf{u} \in \mathbf{G}$ we consider the one-generator subgroup $\mathbf{U} = \{\mathbf{u}^k : k \in \mathbf{Z}\}$ and call $x \in \Gamma\backslash\mathbf{G}$ *generic* for **U** if the closure $\overline{x\mathbf{U}} = x\mathbf{H}$ is homogeneous with respect to **H** and

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} f(x\mathbf{u}^i) = \int_{\Gamma\backslash\mathbf{G}} f\, d\nu_{\mathbf{H}}$$

for all bounded continuous f and $\nu_{\mathbf{H}}$ as above.

Definition 3. Let **G** be a Lie group and **U** a one-parameter or one-generator subgroup of **G**. We say that **U** is *distribution rigid* in **G** if for every lattice Γ in **G** every point $x \in \Gamma\backslash\mathbf{G}$ is generic for **U**.

We prove the following

Theorem B (The Main Theorem). *Every one-parameter or one-generator unipotent subgroup* $\mathbf{U}$ *of a connected* $\mathbf{G}$ *is distribution rigid.*

It is clear that Theorem B implies Theorem A for $\mathbf{U}$. When $\mathbf{G}$ is nilpotent, Theorem B follows from [P]. As to semisimple $\mathbf{G}$ it was shown in [DS] that Theorem B holds for $\mathbf{G} = SL(2, R)$. To the best of my knowledge these are the only cases for which Theorem B was proved.

The notion of a generic point and distribution rigidity can be given for *any* unipotent subgroup $\mathbf{U}$ of $\mathbf{G}$ with averages performed over the regular subsets of $\mathbf{U}$ discussed in [R1] (see also [GE, OW]). It seems plausible that an analog of Theorem B holds for such sets. We show that if $\mathbf{G}$ is nilpotent and $\mathbf{U}$ is connected then Theorem B holds for all regular sets in $\mathbf{U}$.

Now let $\mathbf{U}$ be a unipotent subgroup of a connected $\mathbf{G}$ and $x \in \Gamma\backslash\mathbf{G}$. Suppose that $\overline{x\mathbf{U}}$ is homogeneous with respect to some closed subgroup $\mathbf{H}$ of $\mathbf{G}$. Define

$$\mathbf{H}(x, \mathbf{U}) = \mathbf{H}^0\mathbf{U}$$

where $\mathbf{H}^0$ denotes the identity component of $\mathbf{H}$. It is clear that $\mathbf{H}(x, \mathbf{U})$ is the smallest closed subgroup $\mathbf{K}$ of $\mathbf{G}$ such that $\overline{x\mathbf{U}}$ is homogeneous with respect to $\mathbf{K}$. Define

$$\Phi_x(\mathbf{G}, \Gamma) = \{\mathbf{H}(x, \mathbf{U}) : \mathbf{U} \text{ is a unipotent subgroup of } \mathbf{G}\}.$$

We have

$$\Phi_x(\mathbf{G}, \Gamma) = \mathbf{x}^{-1}\Phi(\mathbf{G}, \Gamma)\mathbf{x}$$

where $\Phi(\mathbf{G}, \Gamma) = \Phi_e(\mathbf{G}, \Gamma)$, $e = \pi(\mathbf{e})$, $\mathbf{x} \in \pi^{-1}\{x\}$. Our proof of Theorems A and B implies the following

Corollary A. (1) *The set* $\Phi(\mathbf{G}, \Gamma)$ *is countable.* (2) $\mathbf{U}$ *acts ergodically on* $(\overline{x\mathbf{U}} = x\mathbf{H}, \nu_\mathbf{H})$, $x \in \Gamma\backslash\mathbf{G}$, *where* $\mathbf{H} = \mathbf{H}(x, \mathbf{U})$.

Corollary A implies the following

Corollary B. *Let* $\mathbf{G}$ *be a connected Lie group and* $\mathbf{U}$ *a Lie subgroup of* $\mathbf{G}$ *of the form* $\mathbf{U} = \bigcup_{i=1}^{\infty} \mathbf{u}_i\mathbf{U}^0$ *where* $\mathbf{u}_i$, $i = 1, 2, \ldots$ *are unipotent in* $\mathbf{G}$ *and* $\mathbf{U}^0$ *is generated by unipotent elements of* $\mathbf{G}$ *contained in* $\mathbf{U}^0$. *Then* $\mathbf{U}$ *is topologically rigid in* $\mathbf{G}$.

Corollary B was conjectured by Margulis in [M, Conjecture 1].

Our proof of Theorem B is totally different form that in [DS] for $G = SL(2, R)$. The central role in the proof of the main theorem is played by Theorem C below proved in [R3]. To state it let us introduce some notations. Let G be a Lie group, Γ a discrete subgroup of G and μ a Borel probability measure on $\Gamma \backslash G$. Define

$$\Lambda = \Lambda(G, \Gamma, \mu) = \{g \in G: \text{the action of } g \text{ on } \Gamma \backslash G \text{ preserves } \mu\}.$$

The set Λ is a closed subgroup of G. The measure μ is called *algebraic* if there exists $x = x(\mu) \in G$ such that $\mu(\pi(x)\Lambda) = 1$. In this case $x\Lambda x^{-1} \cap \Gamma$ is a lattice in $x\Lambda x^{-1}$.

Theorem C [R3, The Main Theorem]. *Let G be a Lie group, Γ a discrete subgroup of G and U a unipotent subgroup of G. Then every ergodic U-invariant Borel probability measure μ on $\Gamma \backslash G$ is algebraic.*

Theorem C and our proof of Corollary B imply the following

Corollary C. *Let G be a connected Lie group, Γ a lattice in G and U a subgroup of G as in Corollary B. Then every ergodic U-invariant Borel probability measure on $\Gamma \backslash G$ is algebraic.*

Here is an outline of the proof of Theorem B for unipotent flows $U = \{u(t) = \exp tu : t \in R\}$. It suffices to prove (*) for every $f \in C_0(X)$, $X = \Gamma \backslash G$ where $C_0(X)$ denotes the Banach space of all real-valued continuous functions on X vanishing at infinity with the supremum norm. We denote by $C_0^*(X)$ the dual of $C_0(x)$. For $x \in \Gamma \backslash G$, $t > 0$ we define $T_{x,t} \in C_0^*(X)$ by

$$T_{x,t}(f) = \left(\int_0^t f(xu(s)) \, ds \right) / t$$

and denote by $M(x, U)$ the set of all limit points of $\{T_{x,t} : t > 0\}$ when $t \uparrow \infty$ in the weak $*$-topology on $C_0^*(X)$. Then each member of $M(x, U)$ is a U-invariant Borel measure on X. We show that $\mu(X) = 1$ for all $\mu \in M(x, U)$. Let $Y_\mu \subset \overline{xU}$ be the support of μ. We consider the ergodic decomposition $\{(C(y), \mu_{C(y)}) : y \in Y_\mu\}$ of the action of U on (Y_μ, μ) and write $\xi_\mu = \{\overline{C(y)} : y \in Y\}$. By Theorem C each $\mu_{C(y)}$ is algebraic. This means that $\overline{C(y)} = y\Lambda_y^0$, where $\Lambda_y = \Lambda(G, \Gamma, \mu_{C(y)})$, $\mu_{C(y)}(y\Lambda_y^0) = 1$. We show that there exists $C = \overline{C(y_0)} \in \xi_\mu$ such that

$$(**) \qquad\qquad \xi_\mu = \{C\} \bmod \mu \quad \text{and} \quad x \in C.$$

This implies that $\overline{x\mathbf{U}} = C$, $\mu(C) = 1$ and μ is $\Lambda^0_{y_0}$-invariant. Since this completely determines $\mu \in M(x, \mathbf{U})$ we obtain $M(x, \mathbf{U}) = \{\mu\}$. This implies distribution rigidity for $\mathbf{U}$ by the definition of $M(x, \mathbf{U})$. To prove $(**)$ we use an analog of the H-property introduced in [R4] (see also [R1, the R-property] and [W, the Ratner property]) and [R2, Theorem 1] which provides a complete description of the ergodic components $C(y)$, $y \in Y_\mu$ when $\mathbf{G}$ is semisimple.

Theorems A and B provide some important number-theoretic applications (see [M] for some of such applications).

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