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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 24, Number 1, January 1991

0273-0979/91 $1.00 + $.25 per page

The Riemann problem and interaction of waves in gas dynamics, by Tung Chang (Tong Zhang) and Ling Hsiao (Ling Xiao). Longman Scientific and Technical (Pitman Monographs No. 41), Essex, 1989, 272 pp. ISBN 0-582-01378-X

Many phenomena involving nonlinear wave motion fit into the mathematical framework of the so-called "hyperbolic systems of conservation laws." These are systems of nonlinear partial differential equations which describe the conservation of certain physical quantities, e.g., mass, momentum, energy, etc. The equations take the form $\operatorname{div} \phi(u) = 0$, where the divergence is with respect to the space-time independent variables, and ϕ is a nonlinear function of the unknown state variable u.

The most mathematically well-understood case is that of one spatial variable, and here the equations take the form of an $n \times n$ system.

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \quad \text{or} \quad \frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0, \tag{1}$$

where $u = u(x, t)$ is an n-vector, and $A(u)$ is the Jacobian matrix of f. The system (1) is called hyperbolic if A has n real eigenvalues $\lambda_i = \lambda_i(u)$, $i = 1, 2, \ldots, n$, and is strictly hyperbolic if the λ_i are all distinct. The λ_i are the wave speeds, or characteristic speeds, and they govern the propagation of infinitesimal disturbances. An important problem is to solve these equations subject to prescribed initial conditions on the state variable u. That is, one is given "initial data" (u at $t = 0$),

$$u(x, 0) = u_0(x), \tag{2}$$

and it is required to solve the "initial-value problem" (1), (2) in the region $t > 0$. The most interesting phenomenon associated with this problem is the necessary occurrence of solution singularities — in fact, smooth f and smooth data u_0 often do not even allow smooth solutions! This is due to the fact that jump discontinuities (shock waves) form spontaneously in the solutions.

In the simplest case of a single equation ($n = 1$), nonlinear waves travelling at different speeds cannot interact, and as a result, the mathematical theory for the problem (1), (2) is in good shape. This is mainly due to the pioneering work of E. Hopf [3] and O. A. Oleinik [7]. As soon as n exceeds one, however, we enter into the field of *systems* of conservation laws, and the subject becomes both richer and far more complicated.

Because of the formidable mathematical difficulties associated with the general initial-value problem (1), (2), much attention has been given to problems having certain special features, like specifying a particular form of f, and/or u_0. As the title indicates, the book under review is concerned with just such things. In fact, the phrase "Riemann problem" is the jargon in the field for initial data u_0 of the following form:

$$u_0(x) = \begin{cases} u_L, & x < 0 \\ u_R, & x > 0, \end{cases} \tag{3}$$

where u_L and u_R are constant n-vectors. Now at first glance, one might be somewhat surprised at this — really? An entire book devoted to this very special problem? The answer is that this very special problem has so far been the most important one in the entire field because its solutions serve as "building blocks" for solving problems with more general initial data. Indeed, the understanding of solutions of the Riemann problem has served to clarify the entire field, and to allow great advances to be made in understanding both qualitative and quantitative questions for much more general problems. Let me give just a few examples. Thus, in Lax's important early study of the Riemann problem [5], one finds for the first time, all of the basic notions in the field. Moreover, Glimm's fundamental paper [1], where the problem (1), (2) is solved for general f by a "random-choice" method, is based on a deep study of Riemann problems and how their solutions interact. The Riemann problem solution also contains the large time asymptotics for solutions for a wide class of initial data u_0, as was shown by Liu [6]. Finally, the Riemann problem has also served as a testing ground for the important area of the numerical analysis for solutions of (1), (2). (Here, however, my own feeling is that many workers have put far too much weight on how one particular difference scheme approximation fares against another one, where the test in both cases is made merely on solutions of the Riemann problem for a *scalar* equation, usually the "Burgers" equation, $u_t + (u^2/2)_x = 0$ — more about this later.)

The Riemann problem (1), (2) actually arises in an important physical problem, first studied by Riemann, which is worth describing. Thus, consider a gas confined to a "long" and "thin" tube (so it can be modelled by a single spatial variable x, $-\infty < x < \infty$), having a thin membrane partitioning the tube into two regions, $x < 0$ and $x > 0$. In this model, the state vector $u = (\rho, E, v)$ is three dimensional, representing, respectively, the density, energy, and velocity of the gas, and u_L and u_R denote two different constant states of the gas at rest ($v_L = 0 = v_R$), on both sides of the membrane. At time $t = 0$, the membrane

$$u_L \mid u_R$$
$$x = 0$$

is broken, and the problem is to describe the ensuing motion of the gas. By specializing (1) to the gas dynamics equations (which describe the three conservation laws of mass, momentum, and energy), one immediately sees that the mathematical description of this problem leads to a Riemann problem. It is well known that much of the early work on nonlinear wave motion was done by Riemann, whose ideas were based on the problem that now bears his name.

Since the equation (1) and the initial data (3) are invariant under the change of scale

$$(x, t) \to (\sigma x, \sigma t),$$

one is motivated to seek solutions of the Riemann problem which are functions of the single variable $\xi = x/t$. Thus, as was shown by Lax [5], the solution will consist of elementary waves, called shock waves, rarefaction waves, or contact discontinuities, and for an $n \times n$ system, the general solution will consist of n waves, separated by wedges in which u takes constant values. For example, in Figure 1, for two equations, there are three wedges and one new constant state u_M. The elementary waves are defined by solutions of either ordinary differential equations or algebraic equations in the state space u.

The solution just sketched is valid for the gas dynamics equations, where the flux function f is "genuinely nonlinear" (in a sense discovered by Lax [5]), and the system is strictly hyperbolic. However, one or both of these conditions may fail to hold in several important systems which arise in chemistry, elasticity, magnetohydrodynamics, and oil reservoir applications, to name but a few. (I should remark that the topic of when only genuine nonlinearity breaks down, but strict hyperbolicity is preserved, is discussed in the book under review.) In the more degenerate cases, the above picture is too simple, and many complications arise. For example, rarefaction waves with embedded shocks may occur, and degenerate wave speeds $\lambda_k = \lambda_{k+1}$ may occur on an open set in u-space. Furthermore, the degeneracy set may occur on a boundary between hyperbolic behavior (all λ_k real) and elliptic behavior (certain λ_k occurring in complex conjugate pairs). Much good work has been done in the last 10 years or so, with a goal toward classifying and understanding the allowable discontinuities in these cases; see [4].

There is another important reason for studying Riemann problems which can now be described; namely, knowing how to solve

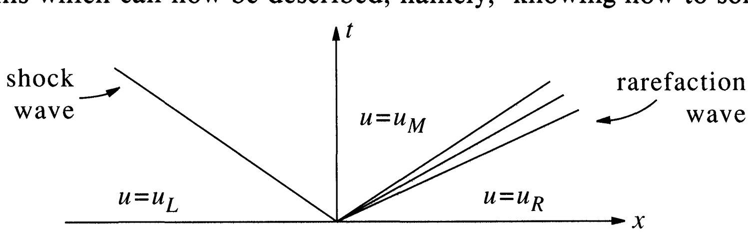

FIGURE 1

Riemann problems allows one to understand the interaction of shock waves. For example, if we take initial data u_0 consisting now of three constant states

$$
u_0(x) = \begin{cases} u_L, & x < a \\ u_M, & a < x < b \\ u_R, & x > b, \end{cases} \tag{4}
$$

where the two corresponding Riemann problems with data (u_L, u_M) and (u_M, u_R) are each resolvable by a single shock wave, then the problem (1), (4) leads to an interaction of shock waves (the overtaking of one shock wave by another), as depicted in Figure 2.

From the picture, it follows that at time $t = t_1$, we are again confronted with a Riemann problem, with data (u_L, u_R). We thus see that if we can resolve Riemann problems, then we can resolve certain interactions of nonlinear waves. It is my opinion that workers studying difference scheme approximations to conservation laws ought to test their methods on problems which involve wave interactions, and not merely on Riemann problems. (Of course, there are some researchers who are already putting these ideas into practice — for example, Glimm's "front-tracking" scheme [2] is practically defined in terms of wave interactions.)

The book under review consists of four chapters. In the first chapter the Riemann problem for a scalar conservation law is considered in great detail for general nonconvex f. There is also included a discussion of Lax's general solution of the Riemann problem for strictly hyperbolic systems, where $|u_L - u_R|$ is sufficiently small. The next chapter consists of a study of a model problem of two equations (insentropic gas dynamics), including a discussion of the interaction of elementary waves. A proof of Glimm's existence theorem is also outlined. An exhaustive discussion of a perturbed Riemann problem is also included. In Chapter 3, the full system of gas dynamics equations is considered. This

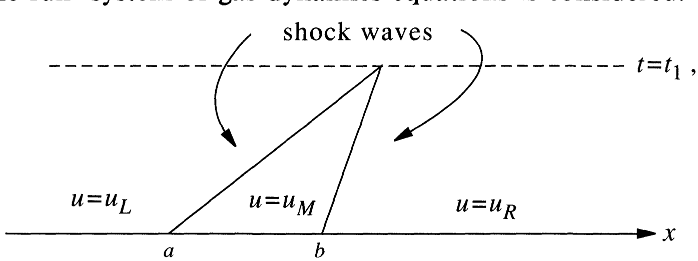

FIGURE 2

chapter is fairly difficult to read, as certain things are unmotivated (for example, the important Hugoniot relation on page 98 is just stated out of the blue!), and the reader is also confronted with the jungle of details in the "non-convex" case. (It would be much better for the nonexpert to first study the less general "convex" case.) Furthermore, I was unable to find a precise statement of the solution of the Riemann problem. The final chapter is concerned with flows in two space dimensions. Here the authors give a discussion of the scalar conservation law, together with some miscellaneous results for systems: overtaking of shocks in steady flow, and planar shock diffraction; furthermore, the problem of Mach and regular reflection is stated.

This book is not for beginners — rather, it is more of a research monograph written for people who already have some understanding of the field. The text has a few places where results are prefaced with "it can be shown that," and proofs are omitted but no references are given. Perhaps the most glaring omission in the book is any mention of the important result of R. Smith [8], who first showed for the Riemann problem in gas dynamics that uniqueness of the "entropy" solution is lost unless the equation of state satisfies some additional condition.

The subject of systems of conservation laws is a difficult one, and any text is a welcome addition to the literature. All in all, the book is a good one and the authors are to be commended for their efforts in making these ideas more widely accessible.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 24, Number 1, January 1991

0273-0979/91 $1.00 + $.25 per page

Modular forms, by T. Miyake. Springer-Verlag, Berlin, New York, 1989, 335 pp., $73.00. ISBN 3-540-50268-8

Modular forms have been studied, *accidently or intentionally*, for about 200 years, beginning seriously with Jacobi and Eisenstein. A key word here is "accidentally": Historically, many peculiar things were discovered and studied in an *ad hoc* fashion; a great number of these are now construed as corollaries of a general phenomenology with the unfortunately unevocative appellations "theory of modular forms" or "theory of automorphic forms." This "underlying phenomenology" is distant from more tangible and elementary issues, and so often seems obscurely technical and tiresomely unmotivated (to the uninitiated, at least).

Because it does provide an underlying pattern, the subject is currently of intense research interest. Either provably or conjecturally, a large fraction of the objects of interest in number theory is intimately related to modular forms. There are also pleasantly surprising connections with many other things: string theory, combinatorics, Kac-Moody algebras, and so on.

To develop a sympathy for the subject, it seems necessary to shift what one believes to be the *primary objects of study*. Because of the efficacy of "the theory of modular forms" as a methodology in number theory, one might study modular forms as fundamental objects, rather than directly consider number fields themselves (for example). To add to the confusion of the novice, there is not a single fixed notion of "modular form": The general idea has many different incarnations, whose common spirit is apparent only after considerable reflection (and proof). Yet, each incarnation of "modular form" has its own utility.

With just a few exceptions, the subject languished for the first half of this century. Its resurrection in the late 1950s and early