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THREE RIGIDITY CRITERIA FOR PSL(2, R)

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STATEMENT OF RESULTS

Let G be $\mathrm{PSL}(2, \mathbf{R})$, the quotient of the group of 2×2 real matrices with determinant one by its two element center, $\{\pm I\}$. By a *lattice subgroup* of G we mean a discrete subgroup such that the space of cosets G/Γ has finite volume. A familiar example of a lattice subgroup is $\mathrm{PSL}(2, \mathbf{Z})$, the subgroup of matrices in $\mathrm{PSL}(2, \mathbf{R})$ with integer entries. Let Γ be an abstract group and let ι_1 and ι_2 be two inclusions of Γ in G, each having a lattice subgroup as its image. We say ι_1 and ι_2 are *equivalent* if there is some (continuous) automorphism α of G so that $\iota_2 = \alpha \circ \iota_1$. This paper describes three closely related criteria for the equivalence of ι_1 and ι_2: one analytic, one representation theoretic, and one geometric.

If G were $\mathrm{PSL}(n, \mathbf{R})$ for some $n > 2$, or indeed if it were any connected simple Lie group with trivial center except for $\mathrm{PSL}(2, \mathbf{R})$, then the Mostow rigidity theorem (see [M1, M2, Ma, P]) would assert that ι_1 and ι_2, as described above, are necessarily equivalent: a given abstract group Γ could be embedded in G as a lattice subgroup in at most one way (up to automorphisms of G). This remarkable theorem is false for $\mathrm{PSL}(2, \mathbf{R})$. Indeed, the

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study of the set of equivalence classes of embeddings of a given Γ as a lattice subgroup of $\mathrm{PSL}(2, \mathbf{R})$ is the main focus of the highly developed Teichmüller theory (see, [Ah, Ga, L, N]). The following results are rigidity theorems for $\mathrm{PSL}(2, \mathbf{R})$, although they have additional hypotheses, as they must.

We say ι_1 and ι_2 are *topologically conjugate* if there is some (orientation-preserving) homeomorphism β of $\mathbf{H}$ such that $\iota_2(\gamma) = \beta \circ \iota_1(\gamma) \circ \beta^{-1}$. Such a β will extend uniquely to a homeomorphism of $\mathbf{R} \cup \infty$, the boundary of $\mathbf{H}$. Moreover, the boundary homeomorphism is completely determined by ι_1 and ι_2, even though the interior homeomorphism is not. If Γ is the fundamental group of a surface, the equivalence classes of embeddings topologically conjugate to a given one, ι, can be identified with points of the Teichmüller space of the surface $R = \mathbf{H}/\iota(\Gamma)$. If R is a Riemann surface of genus g with k punctures, then the associated Teichmüller space can be given the structure of a $3g-3+k$ dimensional complex manifold; in particular, it is uncountable.

$\mathrm{PSL}(2, \mathbf{R})$ acts on the upper half-space, $\mathbf{H}$, by the well-known recipe

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = (az+b)/(cz+d).$$

The action preserves the *hyperbolic* metric $ds^2 = (dx^2 + dy^2)/y^2$, and in fact G is the full group of orientation-preserving isometries of H_+. The analytic criterion uses the function $h : G \to \mathbf{R}^+$ given by $h(g) = \exp(-d(g \cdot i, i))$, where $d(\cdot, \cdot)$ is hyperbolic distance on $\mathbf{H}$. This function belongs to $L^{1+\epsilon}(G)$ for any $\epsilon > 0$ but not to $L^1(G)$. Similarly, if $h_j : \Gamma \to \mathbf{R}^+$ is the composition of h with ι_j, then h_j belongs to $l^{1+\epsilon}(\Gamma)$ for any $\epsilon > 0$ but not to $l^1(\Gamma)$.

Theorem 1. *Fix s, $0 < s < 1$. The inclusions ι_1 and ι_2 are equivalent if and only if*

$$\sum_{\gamma \in \Gamma} h_1^s(\gamma) h_2^{1-s}(\gamma) = +\infty.$$

For the representation theoretic criterion, we let π_1 and π_2 be nontrivial irreducible unitary representations of G (see [K]) which *don't* belong to the discrete series. (That is π_1 and π_2 belong to the principal or complementary spherical series.) According to [C-S], $\pi_j \circ \iota_j$ is an irreducible representation of Γ for $j = 1$ or 2. As usual, two irreducible unitary representations of a group are called *equivalent* if there is a unitary equivalence between the two representation spaces which intertwines the two group actions.

Theorem 2. *The representations* $\pi_1 \circ \iota_1$ *and* $\pi_2 \circ \iota_2$ *of* Γ *are equivalent if and only if the representations* π_1 *and* π_2 *of* G *are equivalent and the inclusions* ι_1 *and* ι_2 *are equivalent.*

This is not stated as a criterion for the equivalence of ι_1 and ι_2, but we obtain one by setting $\pi_1 = \pi_2$. Theorem 2 fails for discrete series representations. If π_1 and π_2 are in the discrete series, then $\pi_1 \circ \iota_1$ and $\pi_2 \circ \iota_2$ are square integrable representations of Γ, hence continuously reducible. Any square integrable representation of Γ is characterized up to unitary equivalence by a single real number, its continuous dimension, and the continuous dimension of $\pi_j \circ \iota_j$ is the product of the formal dimension of π_j and the volume of $G/\iota_j(\Gamma)$ (see [G-H-J, Theorem 3.3.2].) If Γ is torsion free, then it determines the Euler characteristic and thus the volume of $G/\iota_j(\Gamma)$. One can always find a finite index, torsion free subgroup of Γ, so $G/\iota_1(\Gamma)$ and $G/\iota_2(\Gamma)$ have equal volumes. The two representations $\pi_1 \circ \iota_1$ and $\pi_2 \circ \iota_2$ are equivalent if and only if π_1 and π_2 have the same formal dimension.

For the final criterion we assume ι_1 and ι_2 are topologically conjugate and that β is the conjugating homeomorphism. It is known that β is either Möbius or singular depending on whether or not ι_1 and ι_2 are equivalent (e.g., [M1]). See [Ag] for a survey of related results. Let $\dim(E)$ denote the Hausdorff dimension of E.

Theorem 3. *Suppose that* ι_1 *and* ι_2 *are topologically conjugate. Then* ι_1 *and* ι_2 *are inequivalent if and only if there exists* $\delta > 0$ *and* $E \subset \mathbf{R}$ *such that* $\dim(E) \leq 1-\delta$ *and* $\dim(\beta(\mathbf{R}\setminus E)) \leq 1-\delta$.

The complete proofs of these results are contained in [B-S]. We thank the referees for their comments and suggestions.

Outline of the proofs

Theorems 1 and 3 are easy to verify under the hypothesis that ι_1 and ι_2 are equivalent, as is Theorem 2 if we assume also that π_1 and π_2 are equivalent. The case of equivalent ι_j and inequivalent π_j is contained in [C-S]. Thus, in the following discussion we assume that ι_1 and ι_2 are inequivalent. Given that, Theorem 2 is a consequence of Theorem 1 and Theorem 3 is a rather direct consequence of the following sharper version of Theorem 1.

Theorem $1'$. *If* ι_1 *and* ι_2 *are inequivalent and* $0 < s < 1$, *then there is some* $\delta > 0$ *so that*

$$\sum_{\gamma\in\Gamma}(h_1(\gamma)^s h_2(\gamma)^{1-s})^{1-\delta} < +\infty .$$

In sketching the proof of Theorem 2, we treat only the case when π_1 and π_2 are in the principal series but not at its endpoint. Let $\mathscr{H}_j$ be the Hilbert space on which π_j acts, and let $U : \mathscr{H}_2 \to \mathscr{H}_1$ be a presumed unitary equivalence between $\pi_2 \circ \iota_2$ and $\pi_1 \circ \iota_1$. From [C-S] we use the fact that there is a $t_1 > 0$, characteristic of π_1, so that

$$\operatorname*{weak-lim}_{\epsilon\to 0+}(\pi_1 \circ \iota_1)(\epsilon h_1^{1/2+it_1+\epsilon}) = Q ,$$

where Q is a nonzero operator on $\mathscr{H}_1$. (In fact Q is some multiple of orthogonal projection onto the one-dimensional subspace of $\mathscr{H}_1$ fixed by $SO(2, \mathbf{R})$.) On the other hand, for ψ and ψ' chosen from a dense class of vectors in $\mathscr{H}_2$

$$|\langle \pi_2(g)\psi , \psi' \rangle| \le Ch(g)^{1/2} .$$

For such vectors

$$\begin{aligned} |\langle QU\psi , U\psi'\rangle| &= |\lim_{\epsilon\to 0+} \langle (\pi_1 \circ \iota_1)(\epsilon h_1^{1/2+it_1+\epsilon})U\psi , U\psi'\rangle| \\ &= |\lim_{\epsilon\to 0+} \langle (\pi_2 \circ \iota_2)(\epsilon h_1^{1/2+it_1+\epsilon})\psi , \psi'\rangle| \\ &\le \lim_{\epsilon\to 0+} C\epsilon \sum_{\gamma\in\Gamma} |h_1(\gamma)^{1/2+it_1+\epsilon} h_2(\gamma)^{1/2}| \\ &\le \lim_{\epsilon\to 0+} C\epsilon \sum_{\gamma\in\Gamma} h_1(\gamma)^{1/2} h_2(\gamma)^{1/2} . \end{aligned}$$

According to Theorem 1, this limit is zero, so Q is zero, contradicting the original hypothesis that ι_1 and ι_2 were inequivalent.

To demonstrate Theorem 1 we must again suppose that ι_1 and ι_2 are inequivalent and prove that

$$\sum_{\gamma\in\Gamma} h_1(\gamma)^s h_2(\gamma)^{1-s} < +\infty .$$

Let $\mathscr{T}$ be the vertex of a combinatorial tree and fix a *root* vertex $v_0 \in \mathscr{T}$. For any $v \in \mathscr{T}$, let $D(v)$, the *daughters* of v, consist of vertices adjacent to v but further away from the root vertex than v is. A function $f : \mathscr{T} \to \mathbf{R}^+$ is *additive* if

$$\sum_{w\in D(v)} f(w) = f(v) \quad \text{for each } v \in \mathscr{T} .$$

Lemma 1. *Let f_1 and f_2 be additive functions on $\mathscr{T}$. If for some $\rho < 1$,*

$$\sum_{w\in D(v)} f_1(w)^s f_2(w)^{1-s} \le \rho \left(\sum_{w\in D(v)} f_1(w)\right)^s \left(\sum_{w\in D(v)} f_2(w)\right)^{1-s}$$

for each $v \in \mathscr{T}$, then

$$\sum_{v\in\mathscr{T}} f_1(v)^s f_2(v)^{1-s} < +\infty.$$

Indeed, the sum over the nth generation, that is the sum over the vertices at distance n from v_0, is less than or equal to

$$\rho^n f_1(v_0)^s f_2(v_0)^{1-s}.$$

The hypothesis of Lemma 1 is (in a certain quantitative sense) that the vectors $(f_1(w))_{w\in D(v)}$ and $(f_2(w))_{w\in D(v)}$ are uniformly nonproportional as v varies.

One may, it turns out, replace Γ with any subgroup of finite index. Therefore [S] we may assume Γ is either a free group (case of noncompact G/Γ) or the fundamental group of a closed Riemann surface (case of compact G/Γ). Assume for this discussion that Γ is a free group and fix a (necessarily finite) set of generators $(a_l)_{l=1}^{L}$. Give Γ the usual tree structure, saying that γ and γ' are adjacent if and only if $\gamma' = \gamma a_l^{\pm 1}$ for some l.

To first approximation this is the tree to which Lemma 1 applies. What are the additive functions? For any $\gamma \in \Gamma$, let $D_+(\gamma)$ be the set containing the daughters of γ, the daughters of the daughters of γ, and so on. Let the *segments* of γ be

$$S_j(\gamma) = \overline{\{\iota_j(\gamma')\cdot i\,;\ \gamma' \in D_+(\gamma)\}} \cap (\mathbf{R}\cup\{\infty\}) \quad \text{for } j = 1 \text{ or } 2.$$

Each segment $S_j(\gamma)$ is a finite union of intervals on $\mathbf{R}\cup\{\infty\}$. By choosing the basis of Γ correctly, we may assume that each $S_1(\gamma)$ is a single interval; if ι_1 and ι_2 are topologically conjugate, then we may assume as well that each $S_2(\gamma)$ is a single interval. Except for endpoints $S_j(\gamma)$ is the disjoint union of $(S_j(\gamma'))_{\gamma'\in D(\gamma)}$. Define the additive function $L_j(\gamma)$ as the total length of $S_j(\gamma)$. Then $L_j(\gamma) \ge CH_j(\gamma)$ and it suffices to prove

$$\sum_{\gamma\in\Gamma} L_1(\gamma)^s L_2(\gamma)^{1-s} < +\infty.$$

The first modification to the tree is necessary because $S_j(\gamma)$ sometimes contains infinite intervals and so $L_j(\gamma)$ is sometimes infinite. We divide the tree up into finitely many subtrees. Then for each subtree we conjugate ι_1 and ι_2 by elements of G so as to relocate the points at infinity and make all the $L_j(\gamma)$ finite. Next we eliminate certain of the vertices γ of the tree in order to avoid the situation where one of the entries of the vector $(L_1(\gamma'))_{\gamma' \in D(\gamma)}$ is much greater than the others. This situation occurs near parabolic fixed points. Finally, for a certain N (dependent only on the topological situation) we include the vertices in every Nth generation but exclude the rest. Given any two of the remaining vertices we put an edge between them if the path in the original tree from the one to the other passes only through excluded vertices.

Let Γ_N be the set of words of length less than $N/2$ in Γ. Now suppose the series above diverges. Then by Lemma 1 we can find elements $\gamma \in \Gamma$ such that

(1)

$$\sum_{\gamma' \in D(\gamma)} L_1(\gamma')^s L_2(\gamma')^{1-s} \quad \text{and} \quad \left(\sum_{\gamma' \in D(\gamma)} L_1(\gamma') \right)^s \left(\sum_{\gamma' \in D(\gamma)} L_2(\gamma') \right)^{1-s}$$

are as close are desired. Near equality in Hölder's inequality implies near proportionality of the two vectors involved, which implies in this case that the tuples $(L_1(\gamma\gamma'))_{\gamma' \in \Gamma_N}$ and $(L_2(\gamma\gamma'))_{\gamma' \in \Gamma_N}$ are nearly proportional. From each such γ we construct an isometry (orientation-preserving or orientation-reversing) of $\mathbf{H}$ which nearly conjugates ι_1 to ι_2. Indeed this isometry is the composition of $\iota_2(\gamma^{-1})$ with an affine transformation and with $\iota_1(\gamma)$. Now consider a sequence of γ, which gives better and better agreement in (1). The corresponding sequence of isometries has a limit and this limit conjugates ι_1 to ι_2. Thus, the two inclusions are equivalent.

If Γ is the fundamental group of a closed Riemann surface, then the proof is not very different. The one added difficulty is that of finding a useful tree structure on the group. This done, several other technical difficulties disappear, firstly because there are no parabolic elements in Γ and secondly because by the Dehn-Nielsen lemma the two inclusions are topologically conjugate a priori. We use our own (possibly original) solution to the word problem to establish the tree structure. However, any reasonable solution to the word problem might be equally effective.

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