Equation in Unbounded Domains; and Chapter IX, Scattering Problems Depending on a Parameter. Elastic Structure-Fluid Interaction in Unbounded Domains.

All in all, it is a lovely book and long overdue.

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An introduction to the numerical analysis of spectral methods, by Bertrand Mercier. Lecture Notes in Phys., Springer-Verlag, Berlin, New York, 1989, 154 pp., $23.30. ISBN 3-540-51106-7

Spectral methods form a relatively young and vigorously expanding field of numerical analysis. In the last 10 to 15 years, they have been applied to a wide variety of problems of mathematics and engineering. Concomitantly, the theoretical analysis of these calculations has grown and diversified, although, as usual in practical applications, we still compute much more than we can prove. The interested reader can do no better than consult [1], which surveys the state-of-the-art situation in the late 80s, giving ample coverage to both theory and applications.

The basic ingredient of spectral methods is the expansion of the unknown quantities in the series of orthogonal functions; these functions, in turn, result from the solution of a Sturm–Liouville problem. In practice, one considers either Fourier expansions—usually for periodic problems—or expansions in terms of orthogonal polynomials. Among the latter, the Chebyshev polynomials play a distinguished role, as they are amenable to the fast Fourier transform, but also admit more general boundary values than those allowed in Fourier series.

Consider now a typical differential problem $Lu = f$, for the unknown u. After u is replaced by an N-term expansion in the eigenfunctions ϕ_j

$$u_N = \sum_1^N c_j \phi_j ,$$

the operator L is approximated in one of the three following ways:

Galerkin method: the residual must be orthogonal to the subspace spanned by the N eigenfunctions

$$(Lu_N - f, \phi_j) = 0, \qquad 1 \le j \le N.$$

Lanczos τ-method: the residual must be orthogonal to the subspace spanned by $N - p$ eigenfunctions, while satisfying p additional constraints, e.g. boundary conditions.

Collocation: the residual must vanish at N points x_i, $1 \le i \le N$, in the interval of interest.

In general, the Galerkin and τ-methods employ the expansion coefficients c_j in calculations, while collocation uses the value of the function itself. It is certainly more convenient than the other methods: to use the Galerkin method for a term $a(x)u(x)$ one must find the coefficients of au from the coefficients of u and the multiplier a, while collocation merely requires the values $a(x_i)$, $u(x_i)$. For nonlinear problems collocation is also more advantageous. Obviously, the points x_i must be properly chosen—they are nodes of the quadratures associated with the Sturm–Liouville eigensystem, and as such make it possible to exploit special mathematical properties, e.g. orthogonality under summation.

The main reason for using spectral methods is the basic fact that if a C^∞ function is expanded in Sturm–Liouville eigenfunction series, the coefficients—and therefore the error—will decay rapidly. In general, E_N, the error committed by summing only N terms in the infinite series, will satisfy, for all N, an estimate of the form $E_N < C(M)N^{-M}$ for arbitrarily large M. Contrast this with the much larger asymptotic error for finite differences or finite elements, say $E_N < CN^{-2}$ for a second-order method. From the practical point of view, it is seen that fewer variables are needed to discretize spectrally a problem within a given tolerance, and thus the spectral method should produce precise results while saving computer memory and possibly computer work.

The "spectral accuracy" described above is indeed spectacular as long as only smooth functions are considered. Of course, one cannot expect all problems to have smooth solutions, and large classes of nonlinear equations—e.g. the Euler equations of inviscid gas dynamics—will produce discontinuous solutions out of smooth initial data. Thus, spectral accuracy has to be taken with a grain of salt—actually, the treatment of nonsmooth problems is one of the main topics of current research in spectral methods. Consider, for example, the extreme case of discontinuous functions. As is well known, discontinuities will produce the Gibbs phenomenon, and the resulting oscillations may globally destroy the accuracy of the approximation; however, one may actually use the Gibbs phenomenon to pinpoint the location of jumps. A recent paper [2] used this type of discontinuity treatment to perform very accurate, nonoscillatory gas dynamic calculations for flows with shocks.

A different situation occurs when the function to be expanded possesses a few continuous derivatives but is not in C^{∞}—a natural setting for such functions is Sobolev space. This case is extensively studied in the book under review. In fact, the author was the first to point out that estimates may be obtained in Sobolev norms of negative order for periodic solutions of linear hyperbolic equations. The proofs are presented in detail for both Galerkin and collocation methods, and form the bulk of the first part of the book. These results show that even nonsmooth numerical solutions will converge rapidly in the mean; from the practical point of view, this means that by smoothing the data and filtering the wiggly numerical solution, one may recover accurate values in any region where the exact solution is smooth.

The second part of the book presents several problems with orthogonal polynomial discretization. The numerical quadratures of Gauss, Lobatto, and Radau are presented in detail, as they are needed for boundary value treatment and the definition of polynomial norms. These norms (which are defined only for polynomials of degree $\leq N$) are then used to prove stability of the numerical solution of advection and diffusion equations. The results are weaker than those obtained for Fourier approximations, but this is to be expected: polynomial spectral methods seem to be an order of magnitude harder—when proofs are required—than the periodic spectral methods. To mention a recent example, [3] presents a full convergence proof for a nonlinear equation with Fourier expansion; nothing of the kind is as yet available for other expansions. Since many important problems are not periodic, it is clear that the analysis of polynomial spectral methods is, and will remain for some time, a very active field.

Some other topics discussed are time discretization (i.e. after the spatial operators have been replaced by spectral formulas), and the use of the fast Fourier transform for efficiency.

One must bear in mind some of the limitations of the book under review. As it originates from a course of lectures presented in 1981, it is aimed at students rather than researchers and, perforce, cannot contain many important results, obtained later. The presentation mixes sometimes elementary and advanced topics: one early section defines a Hilbert base and exhibits the family $\{\sin nx\}$ while the very next section uses (without proof) the fact that the eigenvectors of a hermitian compact operator in separable Hilbert space form a Hilbert base, and the fact that the solution of

$$u'' = f, \qquad u(0) = u(\pi) = 0,$$

is such an operator. But this uneven level may serve as a challenge for the student, bringing new subjects to his or her attention.

In conclusion, Mercier's lectures form a useful textbook, self-contained in the treatment of spectral methods and thorough in the analysis of simple differential equations and numerical methods. This book will serve as a rigorous introduction to the Hilbert space and Sobolev space techniques for linear problems, leaving the student well prepared for current numerical analysis of spectral methods. Its best use may be as an introduction and theoretical supplement to the survey [1].

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