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Harmonic analysis of spherical functions on real reductive groups, by R. Gangolli and V. S. Varadarajan. Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 101, Springer-Verlag, Berlin, Heidelberg, and New York, 1988, xiv+365 pp., $110.00. ISBN 3-540-18302-7

1

Let $\mathscr{X}$ be a locally compact Hausdorff space endowed with a transitive action of a locally compact group G. Then $\mathscr{X} = G/K$ for a closed subgroup K. If $\mathscr{X}$ also admits an invariant measure, then G acts on $L^2(\mathscr{X})$ by unitary transformations by the formula

$$L_{\mathscr{X}}(g)f(x) = f(g^{-1}x). \tag{1.1}$$

The study of the decomposition of this representation into a "direct integral" of irreducible components is usually known as harmonic analysis on homogeneous spaces.

Assume that $\mathscr{X} = G/K$ is Riemannian symmetric. A special role is played by $C_c(G//K)$, the space of continuous compactly supported functions on G which are K-invariant under the regular representation $(g_1, g_2) \cdot f(x) = f(g_1^{-1}xg_2)$. Gel'fand [Ge], observed that under convolution, $L^1(G//K)$ is an abelian Banach algebra. Then every element s in the spectrum $\Sigma(G//K)$ can be realized, when restricted to $\mathscr{C}_c(G//K)$, as integration against a unique bounded function ϕ_s, called an elementary spherical function. If we write

$$\mathscr{S}(f)(s) = \int_G f(x)\phi_s(x)\,dx, \tag{1.2}$$

then for an appropriate subclass of functions in $L^2(G//K)$, there exists an essentially unique measure $d\omega$ on Σ, the spectrum of $L^1(G//K)$ on $L^1(G/K)$, such that

$$\begin{aligned} \int_G |f(x)|^2\,dx &= \int_\Sigma |\mathscr{S}(f)(s)|^2\,d\omega(s), \\ f(x) &= \int_\Sigma \mathscr{S}(f)(s)\overline{\phi_s(x)}\,d\omega(s). \end{aligned} \tag{1.3}$$

This is known as the Plancherel formula and Fourier inversion.

One would also like to have an explicit formula for $d\omega$. There is an additional action of the left-invariant differential operators $\mathscr{D}(\mathscr{X})$ on the dense subspace C_c^∞. Then the elementary spherical functions are just K-bi-invariant eigenfunctions for $\mathscr{D}(\mathscr{X})$ normalized to take the value 1 at e. For example, in the case of $\mathbb{R}^n$ and $\mathbb{T}^n$, $\mathscr{D}(\mathscr{X})$ are just the constant coefficient operators. Their eigenfunctions are $e^{i<\nu,x>}$ with ν integral in the case of $\mathbb{T}^n$. These are also the characters of the irreducible finite-dimensional representations. Then (1.2) is just the usual Fourier transform and (1.3) is the classical Plancherel formula and Fourier inversion. For a compact group G, the Peter-Weyl theorem implies that

$$f(x) = \sum_{\pi\in\hat{G}} \dim(\pi)\overline{\operatorname{tr}\pi((1,x)\cdot f)}, \tag{1.4}$$

where

$$\operatorname{tr}\pi(f) = \int_G f(g)\operatorname{tr}\pi(g)\,dg. \tag{1.5}$$

If f is invariant for the action on the right for a subgroup K, then $\operatorname{tr}\pi(f)$ will be 0 unless $\pi\in\hat{G}^K$, the set of equivalence classes of irreducible representations that contain nontrivial K-fixed vectors.

For a compact Riemannian space, the spherical functions can be written as

$$\phi_\pi(x) = \int_K \operatorname{tr}\pi(xk)\,dk. \tag{1.6}$$

Thus in this case, to get an explicit formula we can write

$$f(x) = \int_K f(xk)\, dk, \tag{1.7}$$

plug into (1.4), and integrate over K. The Schur orthogonality relations imply that

$$f(x) = \sum_{\pi \in \hat{G}^K} \mathcal{S}(f)(\pi)\overline{\phi_\pi(x)}. \tag{1.8}$$

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The case of noncompact Riemannian spaces is much more complicated; for one thing the analog of the Peter-Weyl theorem would be Fourier inversion for general functions. We illustrate how the general theory for obtaining $d\omega$ looks like in the case of $G = SL(2, \mathbb{R})$ and $K = SO(2)$. Then $\mathcal{X}$ can be identified with the upper half-plane

$$\mathcal{H} = \{z \in \mathbb{C} | z = x + iy,\ y > 0\}, \tag{2.1}$$

with $SL(2, \mathbb{R})$ acting by conformal transformations. Then $C(G//K)$ can be identified with functions on the upper half-plane which are constant on circles going through iy, i/y centered on the imaginary axis. $\mathcal{D}(\mathcal{X})$ is a polynomial algebra in the Laplace operator

$$\Omega = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right). \tag{2.2}$$

Since the functions are bi-invariant under K, it is enough to consider the radial component of this operator along the imaginary axis. On the group G this comes down to decomposing $g = r(\theta_1)a(t)r(\theta_2)$ and using "polar coordinates" (θ_1, t, θ_2), where

$$r(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \qquad a(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}. \tag{2.3}$$

In t, the eigenvalue problem for determining the elementary spherical functions becomes

$$\frac{d^2\phi}{dt^2} + 2\coth 2t \frac{d\phi}{dt} = -(s^2 + 1)\phi. \tag{2.4}$$

K-bi-invariant functions are determined by their restriction to $A = \{a(t)\}$ and the Haar measure decomposes into

$$dx = J(t) d\theta_1 \cdot dt \cdot d\theta_2, \tag{2.5}$$

with $J(t) = \text{const}\,|\sinh(2t)|$. Thus, $f(a(t))J^{1/2}(t)$ is in $L^2(\mathbb{R})$ if and only if $f \in L^2(G//K)$. Then $\psi_s(t) = J^{1/2}\phi_s(a(t))$ is a solution of the differential equation of the form

$$\left[\frac{d^2}{dt^2} - q(t)\right]\psi_s(t) = -s^2\psi(t). \tag{2.6}$$

The spectral theory of ordinary differential operators of this type, [We, Ko, and CL, pages 251–256], suggests that the spectral measure is determined by the asymptotic behavior of ψ_s as $t \longrightarrow \infty$. For $\mathrm{Re}(is) > 0$ we get

$$c(s) = \lim_{t\longrightarrow\infty} e^{(-is+1)t}\phi_s(a(t)) = \pi^{-1/2}\frac{\Gamma(\frac{1}{2}is)}{\Gamma(\frac{1}{2}(is+1))}. \tag{2.7}$$

The aforementioned theory also suggests

$$d\omega(s) = |c(s)|^{-2}ds. \tag{2.8}$$

This turns out to be indeed the case; in fact properties of the Gamma function give

$$|c(s)|^{-2} = \frac{s\pi}{2}\tanh(\frac{s\pi}{2}), \tag{2.9}$$

and the inversion formula

$$f(a(t)) = \frac{1}{2\pi^2}\int_{\mathbb{R}} \mathscr{S}(f)(s)\overline{\phi_s(a(t))}|c(s)|^{-2}\,ds. \tag{2.10}$$

(The constant in front is dependent on the normalization of the Haar measure.)

In the general case, $\mathscr{D}(\mathscr{X})$ gives rise to a system of partial differential equations; so the spectral theory does not apply. Nevertheless it is possible to carry out the analysis involved. The spherical functions and their asymptotics are determined. An inverse to $\mathscr{S}$ would have to have the formula

$$(\mathscr{J}a)(x) = \int_{\hat{A}} a(s)\overline{\phi_s(x)}|c(s)|^{-2}\,ds, \tag{2.11}$$

where A is a maximal isotropic torus in G. The integral makes sense for a certain subspace of functions, invariant under the canonical action of $W = \mathrm{Norm}_G(A)/A$. The fact that $\mathscr{S}\cdot\mathscr{J} = \mathrm{Id}$ is somewhat formal and was established by Harish-Chandra [HC]. The fact that $\mathscr{J}\cdot\mathscr{S} = \mathrm{Id}$ is much more difficult. Harish-Chandra was able to do this only by developing a general Plancherel formula and Fourier inversion.

A proof that just uses the theory of spherical functions also exists and was established by Rosenberg using results of Helgason and Gangolli. This is done by analyzing the so-called Abel transform on the space $\mathscr{C}_c^\infty(G//K)$ and its relation to $\mathscr{S}$.

3

The book by Gangolli and Varadarajan gives an account of this theory for semisimple symmetric spaces. The core of the book exposes ideas in the work of Harish-Chandra on asymptotics of elementary spherical functions viewed as eigenfunctions for $\mathscr{D}(\mathscr{X})$ (Chapters 4 and 5). The main result, which is the Plancherel formula and Fourier inversion, is in Chapter 6. In addition to Harish-Chandra's proof that $\mathscr{J} \cdot \mathscr{S} = \mathrm{Id}$, the method of Helgason, Gangolli, and Rosenberg is also given, so that the proofs are complete.

Chapter 1 gives an overview of the abstract Plancherel formula for the case of spherical functions. Chapters 2 and 3 give the basic more elementary results needed for Chapters 4 and 5. Chapter 6 contains additional results on L^p with $p \neq 2$.

The style of the book is very clear and essentially complete details are given, except for the expository parts, where references are cited. Ample discussions after each chapter trace the history of the subject starting with the work of Gel'fand and Harish-Chandra in the early fifties and continuing with the contributions of Kostant, Helgason, the two authors themselves, and many others. It is an excellent survey of the theory of spherical functions for Riemannian symmetric spaces.

As far as considering it as a textbook for a graduate course, I think it would only be suitable for an advanced course, possibly a reading course only. For one thing, the techniques used come from many different fields—partial differential equations, differential geometry, Lie groups and Lie algebras, functional analysis, and so on. In this respect, I would highly recommend Chapters 1 and 2 for the expository survey of representation theory, spectral theory of Banach algebras, and just basic structure of Lie groups. In a subject as technical as this, it is important not to lose sight of the main ideas.

Many of the topics in this book are also treated in [He]. His mathematical style is very different and I found it well worthwhile to compare treatments of the same topics while preparing this review. For instance, many more examples are treated in detail. This I think, makes it more suitable for a less-advanced graduate course. I also want to draw attention to the inversion formula for functions that are K-invariant only on the right (Theorem 4.2 of the Introduction for $SL(2, \mathbf{R})$). This is related to the Poisson representation of harmonic functions on the unit disc. Helgason's conjecture which generalizes this to an arbitrary symmetric space has received considerable attention ([KK] as well as [Sch, Wa]).

As far as harmonic analysis for more general spaces of functions on reductive Lie groups, Harish-Chandra obtained a Plancherel formula and Fourier inversion formula for $L^2(G)$ for an essentially arbitrary reductive group. In turn this was generalized to arbitrary algebraic groups as well as some classes of nonalgebraic groups by Duflo [D] and the references within. Other developments in harmonic analysis on reductive Lie groups in the last 10 years have been in the direction of considering the case of an affine symmetric space (K is no longer compact, but rather a real form of a maximal compact subgroup for a different real form of G). The references [F, O, S] provide an incomplete list of papers.

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Binary quadratic forms, classical theory and applications, by D. A. Buell. Springer-Verlag, New York, Berlin, 1989, 247 pp., $35.00.

As the subtitle indicates, this book was written with the intention of alerting the computer-minded reader to the possibility of applying some part of the theory of binary quadratic forms to various problems.

In 1801 Gauss laid the foundation of the arithmetic of the forms $f(x, y) = ax^2+2bxy+cy^2$ in sections 153–335 of the *Disquitiones Arithmeticae*. By 1850 the theory of algebraic numbers, the theory of ideals and the theory of class groups were beginning to emerge. This forced the rewriting of Gauss theory using the Eisenstein form $f(x, y) = ax^2 + bxy + cy^2 = (a, b, c)$, with the discriminant $\Delta = b^2 = 4ac$. This revised Gauss theory is what the author describes as the classical theory.

On page 2, three questions are proposed:

(a) What integers can be represented by a given form?
(b) What forms can represent a given integer?
(c) If a form represents an integer, how many representations exist and how may they all be found?

These questions are answered on pages 74–75 by six theorems. The reader is thus required to read four chapters, whose titles are Elementary Concepts, Reduction of Positive Definite Forms, Indefinite Forms, and The Class Group, to prove these theorems.