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Foliations on Riemannian manifolds, by Philippe Tondeur. Universitext, Springer-Verlag, Berlin, Heidelberg, London, New York, Paris, Tokyo, 1988, 247 pp. $29.80. ISBN 0-387-96707-9 and ISBN 3-540-96707-9. *Riemannian foliations,* by Pierre Molino, (Translated by Grant Cairns). Progress in Mathematics, vol. 73, Birkhäuser, Boston, Basel, 1986, 339 pp. $44.00. ISBN 0-8176-3370-7 and ISBN 3-7643-3370-7

Differential geometry can be described as the study of n-manifolds M having geometric structures. This ceases to be tautological when we expand on the meaning of a "geometric structure." One is given a Lie subgroup $G \subset \mathrm{Gl}(n, \mathbb{R})$, together with a coordinate atlas $\{U_\alpha, x_\alpha\}_{\alpha\in\mathfrak{A}}$, such that, on $U_\alpha \cap U_\beta$, the Jacobian $J(x_\alpha \circ x_\beta^{-1})$ of the change of coordinates is G-valued. This geometric structure, also called a G-structure, defines a reduction of the principal frame bundle $F(M)$ to the group G. Such a reduction, whether or not it comes from a G-structure, is called an "infinitesimal G-structure." If it does arise from a G-structure on M, the infinitesimal G-structure is said to be integrable.

A primary example is Riemannian geometry. This is the study of a manifold with a smoothly varying, positive definite inner product (a Riemannian metric) on its tangent spaces. In the above terminology, a Riemannian manifold is an n-manifold M, equipped with an infinitesimal $O(n)$-structure. In this case, the Riemann curvature tensor is the obstruction to integrability, so an integrable Riemannian metric (an $O(n)$-structure) is a euclidean metric on M. All manifolds admit Riemannian metrics, but very few can support euclidean geometry.

A k-dimensional foliation $\mathscr{F}$ of an n-manifold M is another example of an integrable geometric structure on M. Here the structure group G_k is the subgroup of $\mathrm{Gl}(n, \mathbb{R})$ having lower left $(n-k) \times k$ block identically zero. The coordinate atlas can be described as $\{U_\alpha, x_\alpha, y_\alpha\}_{\alpha\in\mathfrak{A}}$, where

$$x_\alpha = (x_\alpha^1, x_\alpha^2, \dots, x_\alpha^k)$$
$$y_\alpha = (y_\alpha^1, y_\alpha^2, \dots, y_\alpha^{n-k}),$$

and, on $U_\alpha \cap U_\beta$,

$$x_\alpha = x_\alpha(x_\beta, y_\beta)$$
$$y_\alpha = y_\alpha(y_\beta),$$

$\forall\alpha, \beta \in \mathfrak{A}$. Every *infinitesimal* G_k-structure is a k-plane subbundle of $T(M)$, called a k-plane distribution on M, and the integrability condition is given by the well-known Frobenius theorem. The k-dimensional level sets in U_α of y_α are local integral manifolds to the distribution and can be assembled into maximal connected integral manifolds, called the "leaves" of the foliation $\mathscr{F}$.

The form of the transformations $y_\alpha(y_\beta)$, defined on $U_\alpha \cap U_\beta$, suggests the presence of a manifold of dimension $q = n - k$ for which the y_α's serve as local coordinates. If the leaves of $\mathscr{F}$ are the level sets of a submersion $f : M \to N^q$, this is exactly right. In this case, N is viewed as the space of leaves and the y_α's provide local coordinates. Generally, the quotient space $M/\mathscr{F}$ is far too pathological to be a manifold, but the y_α's can still be thought of, with care, as local coordinates in a "model" manifold N^q for $M/\mathscr{F}$. More precisely, one can view the coordinate functions y_α as submersions of U_α onto an open subset of $\mathbb{R}^q$ or, indeed, onto a coordinate neighborhood in any fixed manifold N of dimension q. The coordinate changes are then viewed as diffeomorphisms

$$g_{\alpha\beta} : y_\beta(U_\alpha \cap U_\beta) \to y_\alpha(U_\alpha \cap U_\beta)$$

such that $g_{\alpha\beta} \circ y_\beta = y_\alpha$. The system $\{U_\alpha, y_\alpha, g_{\alpha\beta}, N^q\}_{\alpha,\beta\in\mathfrak{A}}$ is a "structure cocycle" for the foliation and it generates a pseudogroup on N, called the holonomy pseudogroup for the foliation (although it really depends on the choice of cocycle).

These considerations give rise to a notion of "transverse geometric structures" or "holonomy invariant geometric structures" on the foliation. Such a structure is simply a geometric structure (infinitesimal or integrable) on N which is preserved by the holonomy pseudogroup.

An example is the notion of a transversely Riemannian foliation, which is the primary concern in the book of Pierre Molino and figures prominently in that of Philippe Tondeur. The foliation is transversely Riemannian if the structure cocycle can be chosen so that N is a Riemannian manifold and the transition functions $g_{\alpha\beta}$ are local isometries. In a similar spirit, a foliation is transversely Lie if N can be taken to be a Lie group G and each $g_{\alpha\beta}$ to be left translation by a group element. The Lie group G supports a left-invariant Riemannian metric, so a Lie foliation is also Riemannian. Similarly, the foliation is transversely homogeneous if the model manifold can be taken to be a homogeneous space G/K and each $g_{\alpha\beta}$ is translation by an element of G. Again such a foliation is Riemannian, provided that K is compact. More generally, a Riemannian foliation is *locally* homogeneous if the local Riemannian manifolds $f_\alpha(U_\alpha)$, $\alpha \in \mathfrak{A}$, all come equipped with a transitive Lie algebra of local Killing fields. Riemannian foliations with dense leaves are locally homogeneous. Another special type of Riemannian foliation is an e-foliation. Here one is given an absolute parallelism on N which is preserved by the $g_{\alpha\beta}$'s. One defines the metric on N so that the fields defining the parallelism become orthonormal. The Lie foliations are a special case of e-foliations. All of these examples play important roles in the structure theory of Riemannian foliations.

It is useful to compare the problem of finding transverse, infinitesimal G-structures in a foliated manifold $(M, \mathcal{F})$ to the classical problem of finding infinitesimal G-structures on a manifold M. The classical problem reduces to a problem in homotopy theory, namely, that of finding a (smooth) section of the $\mathrm{Gl}(n, \mathbb{R})/G$-bundle associated to the principal frame bundle. In the case that $G = O(n)$, the fiber $\mathrm{Gl}(n, \mathbb{R})/O(n)$ is contractible and the section always exists. This is the homotopy-theoretic reason why every manifold has a Riemannian metric. By contrast, it is difficult for a foliated manifold to have a transverse Riemannian structure. Such a structure is, indeed, a reduction of the normal frame bundle to $G = O(q)$, but this reduction must be invariant under the locally absolute parallelism along leaves defined by the foliated structure. In general, transverse, infinitesimal G-structures are G-reductions of the normal frame bundle, invariant under the locally absolute parallelism. We describe this parallelism.

If $T(\mathcal{F}) \subset T(M)$ is the tangent bundle to the foliation, then the q-plane bundle $Q = T(M)/T(\mathcal{F})$ is called the normal bundle and its associated frame bundle $F(Q)$ is called the normal frame bundle of $\mathcal{F}$. The differential dy_α of the local submersion $y_\alpha : U_\alpha \to N$ submerges $F(Q)|U_\alpha$ into the frame bundle of N and is a morphism of principal $\mathrm{Gl}(q, \mathbb{R})$-bundles. A section of $F(Q)|U_\alpha$ is parallel (along the leaves of $\mathcal{F}|U_\alpha$) if dy_α carries it to a well-defined section of $F(N)|y_\alpha(U_\alpha)$. In U_α this is an absolute parallelism along the leaves of $\mathscr{F}|U_\alpha$ and it defines a locally absolute parallelism along leaves globally. This parallel transport along leaves can also be defined by the "Bott connection," a connection on $F(Q)$ whose curvature vanishes in the leaf directions. Normal vector fields which are parallel along leaves are called "basic" fields, and it is an easy matter to introduce corresponding notions of basic forms and basic tensors.

In the case of a transversely Riemannian foliation, one can see that invariance of the metric under this parallelism requires that the leaves stay a fixed distance apart as one moves along them simultaneously. In general, invariance under the parallelism imposes similarly severe restrictions as to which foliations can support a transverse G-structure. We are no longer dealing with a problem in classical homotopy theory.

In codimension one, a transversely orientable foliation is Riemannian if and only if it is defined by a closed, nonsingular 1-form. Such foliations were studied in the early 1950s by Georges Reeb [Rb], who did not stress the Riemannian aspect. The study of Riemannian foliations, as such, begins in 1959 with Bruce Reinhart's work [Rt1] on foliations with a bundlelike metric. A "bundlelike metric" is just a metric on M under which Q is identified with $T(\mathscr{F})^\perp$ and the metric so induced on Q is invariant under the parallelism along leaves. In [Rt1, Rt2], Reinhart obtained important properties of bundlelike metrics. Among these are (1) the fact that geodesics perpendicular to some leaf at a given point are perpendicular, at each of their points, to the leaf through that point, and (2) the fact that all leaves have the same universal cover (if M is connected). It was left to Molino, however, in the late 1970s and early 1980s, to develop a comprehensive structure theory for these foliations [M1–M4].

For simplicity, in what follows we assume that M is compact. Since $O(q)$ is a compact group, it follows that the orthonormal frame bundle $O(Q)$, associated to the bundlelike metric, has compact total space.

The key idea in Molino's theory is to use the locally absolute parallel transport of orthonormal frames along leaves to lift $\mathscr{F}$ to a foliation $\widehat{\mathscr{F}}$ of the orthonormal frame bundle $O(Q)$. Since the parallelism is only *locally* absolute, the leaves of $\widehat{\mathscr{F}}$ will generally be many-to-one covers of the leaves of $\mathscr{F}$. The orthonormal frame bundle $O(N)$ of the model manifold has an absolute parallelism that is preserved by isometries of N. Since $O(N)$ serves as a model for the leaf space of $\widehat{\mathscr{F}}$, it follows easily that $\widehat{\mathscr{F}}$ is an e-foliation, so the next order of business is to describe the structure of e-foliations.

The reviewer studied e-foliations of codimension two in the early 1970s [C], establishing some properties that were analogous to the case of codimension one [Rb]. This work was generalized and greatly simplified by Molino [M4] to include arbitrary codimension. The main facts from Molino's theory for e-foliated $(M, \mathscr{F})$ are that (1) the closures $\overline{L}$ of the leaves L are, themselves, manifolds and smoothly fiber the underlying manifold, and (2) the induced foliation $\overline{\mathscr{F}} = \mathscr{F}|\overline{L}$ is a Lie foliation, independent, up to isomorphism, of the choice of fiber $\overline{L}$. The corresponding simply connected Lie group G and its Lie algebra $\mathfrak{G}$ are called the structural group and the structural algebra, respectively, of $\mathscr{F}$.

When $\mathscr{F}$ is a Riemannian foliation, then $\widehat{\mathscr{F}}$ has leaf closures fibering $O(Q)$ and an associated structural group G (respectively, structural algebra $\mathfrak{G}$) which is called the structural group (respectively, algebra) of the Riemannian foliation $\mathscr{F}$. While the leaf closures for $\widehat{\mathscr{F}}$ are mutually diffeomorphic, their positions relative to the $O(q)$-fibers are not generally uniform; hence their projections into M are submanifolds of varying diffeomorphism types, even of varying dimensions. These submanifolds are exactly the leaf closures for $\mathscr{F}$, they partition M, and, for each leaf L of $\mathscr{F}$, the induced foliation $\overline{\mathscr{F}} = \mathscr{F}|\overline{L}$ is a locally homogeneous foliation.

The reader who would like to pursue this theory in detail is strongly advised to study the two books under review. The book of Molino is an organized account of this structure theory by the man who developed it. The book is entirely self-contained, supposing only a working knowledge of differential geometry. As a delightful bonus, it contains five appendices (by Yves Carrière, Vlad Sergiescu, Grant Cairns, Eliane Salem, and Etienne Ghys), treating related developments and future directions for research. The book of Tondeur has a broader scope and a different purpose. It is the text of a course, given by the author in 1986 at the University of Illinois, on a wide variety of interactions between Riemannian geometry and the theory of foliations. Of special note here is a treatment of the Hodge theory for Riemannian foliations, a subject developed by Ph. Tondeur and F. Kamber in this country [KT1–KT4] and by G. Hector, A. El Kacimi, and V. Sergiescu in France [EH1, EH2, EHS]. Also of note is an enormous bibliography (64 pages) on foliation theory.

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