the 1978 Conference at Stony Brook (I. Kra and B. Maskit, eds.), Ann. of Math. Stud., vol. 97, Princeton Univ. Press, Princeton, N.J., pp. 465–496.

[St] D. Sullivan and W. Thurston, *On manifolds with canonical coordinates*, Enseign. Math. (2) **29** (1983), 15–25.

[T] W. Thurston, *The geometry and topology of 3-manifolds*, Princeton Univ. lecture notes, 1979.

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Real analysis and probability, by R. M. Dudley. Wadsworth & Brooks/Cole Math Series, $52.95. ISBN 0-534-10050-3

There exist many books on each of the areas of real analysis and probability, including some which attempt to treat both subjects in the same treatise. Therefore, one may ask for a compelling reason to publish yet another work on this conjunction of well-established subjects.

Real analysis at the graduate level traditionally consists of measure and integration theory with an introduction to functional analysis. The prevailing tendency has been to treat these topics at an abstract level, with little or no historical commentary and almost no explicit reference to either the motivation or the applications of the material. At the same time we are told that measure theory provides a rigorous foundation for probability theory, while functional analysis has its origins in the theory of integral equations and is central to the modern theory of partial differential equations, among other things. For some students these connections might bring the subject more to life, but traditional approaches have opted for the path of efficient pedagogy, leaving the student to fill in the gaps for himself or herself.

In the case of probability theory, the development of measure and integration theory is long overdue. The earliest form of the weak law of large numbers was proved by Jakob Bernoulli [Be] in 1713; the first version of the central limit theorem was published by Abraham de Moivre [M] (at the age of 66) in 1733, exactly 200 years prior to the measure-theoretic framework which Andre Kolmogorov [K] introduced in 1933. Perhaps the first person to have done probability in the context of measure theory was Norbert Wiener [W] who in 1923 defined the Brownian motion process of Einstein and Smoluchowski (the Wiener process) directly in terms of Lebesgue measure on the unit interval [0,1].

These comments may serve to remind the reader that advanced real analysis is founded on measure theory and that probability theory can be considered honest mathematics only inasmuch as it is based on a careful treatment of measure theory. Most graduate texts on probability theory either assume a measure-theoretic prerequisite (Lebesgue integration or a more abstract version thereof) or give a very skimpy coverage of the subject. Most books on real analysis make only passing reference to probability, encapsulated in the memorable put-down "It suffices to let the basic measure space have total mass 1." A fundamental strong point of Dudley's book is that we are led through a careful course in measure theoretic-analysis before the introduction of any probabilistic concepts. Rather than provide the "shortest possible route" to the rigorous study of probability theory, the book does full justice to the best traditions of real analysis: careful proofs of the sharpest versions of theorems, with attendant consideration of relevant pathologies.

Pathology is the price of abstraction in real analysis (and perhaps more generally). In elementary analysis we define the notions of continuity and compactness on the real line. These yield the very useful results that a real-valued continuous function on a compact set is bounded and attains its extreme values and that the continuous image of a compact (resp. connected) set is again compact (resp. connected). These immediately justify many statements made in elementary calculus, where one has not developed the logical machinery to deal with these basic issues. At the same time, the general notions of continuity and compactness contain some examples to shock the beginning student: Weierstrass's example of a continuous and nowhere differentiable function challenges our intuitive sense of continuity. Cantor's example of an uncountable compact set in $[0, 1]$—whose complement consists of disjoint intervals whose lengths sum to 1—is basic motivation for elementary measure theory. To find a context for the Weierstrass example, we define the notion of "bounded variation" and eventually prove Lebesgue's theorem that a function of bounded variation possesses a derivative almost everywhere. At the same time we note the existence of Cantor's continuous function of bounded variation, whose derivative fails to exist precisely on Cantor's uncountable set of measure zero.

The realm of advanced measure theory has a corresponding plethora of counterexamples and positive results. Some of these revolve around the notions of compactness and metrizability in general measure spaces. In the 1950s and 1960s it was considered essential to develop measure theory on general locally compact spaces which are not necessarily metrizable. A host of examples of such spaces is provided by the Tychonoff theorem. One can consult the books of Halmos [H], Bourbaki [Bo], and Segal and Kunze [S-K] for such developments, culminating in the construction of Haar measure on general locally compact groups. On the other hand, measure theory on locally compact spaces can lead to some unpleasant counterexamples. A characteristic feature of elementary measure theory is its stability under countable operations: The sequential limit or supremum of a sequence of extended real-valued measurable functions is again a measurable function. The same holds for functions with values in an arbitrary metric space. However if we allow functions with values in a nonmetric space, the situation is less pleasant: There exists a sequence of Borel-measurable functions from $[0, 1]$ to $[0, 1]^{[0,1]}$ whose limit f is not even Lebesgue measurable: There exists an open set $W \subset [0, 1]^{[0,1]}$ such that $f^{-1}(W)$ is not a Lebesgue measurable set in $[0, 1]$.

More serious counterexamples arise when one moves to study stochastic processes, defined as the study of measures on the compact space $\Omega = R^T$, where R is the two-point compactification of the real number line and T is an arbitrary index set. Compactness of Ω follows from Tychonoff's theorem, but other good properties may fail. If T is a topological space, the set of "Borel paths" consists of those Borel measurable $f: T \to R$ naturally viewed as a subset of Ω. In case $T = [0, 1]$ and the measure is defined by a suitable completion of Wiener's measure, then the Borel paths form a measurable set with measure 1. But if T is a separable Hilbert space and the measure is canonically defined by the "isonormal process" in terms of the inner product, then the Borel paths have inner measure zero and outer measure 1, hence nonmeasurable for any completed measure. For this example the function $T \times \Omega \ni (t, \omega) \to f(t, \omega) \in R$ is not measurable for the completed product measure. This contribution of Dudley answers a question originally posed by Doob and Kakutani in 1947 [D].

Dudley's book divides approximately evenly, with 194 pages on real analysis and 186 pages on probability theory, which we enumerate for the record: Chapter 1—Foundations, set theory; Chapter 2—General topology; Chapter 3—Measures; Chapter 4—Integration; Chapter 5— L^p spaces and introduction to functional analysis; Chapter 6—Convex sets and duality of normed spaces; Chapter 7—Measure, topology, and differentiation; Chapter 8—Introduction to probability theory; Chapter 9—Convergence of laws and central limit theorems; Chapter 10—Conditional expectations and martingales; Chapter 11—Convergence of laws on separable metric spaces; Chapter 12—Stochastic processes; Chapter 13—Measurability: Borel isomorphism and analytic sets; Appendixes A-E; author index, and subject index.

Each chapter has a wonderful set of historical notes, together with a multitude of references to both the early and the contemporary literature of the subject(s). Altogether we counted 604 separate bibliographical items, although some items are repeated from one chapter to another. Dudley has taken great pains to maintain high scholarly standards by indicating which references he has seen in the original and which were simply discussed in secondary sources but not seen directly.

The book contains 732 problems of varying difficulty, distributed approximately uniformly among the 75 subsections of the book. Many of these come with hints which are intended to minimize frustration, but are not guaranteed to render all of the mathematics painless. In comparison to "standard textbook" accounts of real analysis and probability theory, we find here new treatments of the completion of metric spaces, Daniell-Stone integration theory, the strong law of large numbers, the (Doob) martingale convergence theorem, the (Kingman) subadditive ergodic theorem, and the (Hartman-Wintner) law of the iterated logarithm. Many of these occur in the 23 starred sections, these being extensions beyond the minimal syllabus for this one-year graduate course.

In conclusion, let us emphasize that Dudley's book is a splendid success on two counts. As a work on real analysis it provides a thorough and up-to-date course which is well motivated and purposeful. As a work on probability theory it is mathematically complete and self-contained. The main body of text makes no ad hoc reference to books or papers for special facts; all references to the literature occur in the historical notes at the end of each chapter.

Finally, we remark that a number of new graduate texts on measure theoretic probability are now being written, soon to appear. The prior appearance of Dudley's book is certain to define a new standard of rigor and completeness for the decade of the 1990s.

REFERENCES

[Be] J. Bernoulli, *Ars conjectandi*, Basel, 1713.

[Bo] N. Bourbaki, *Integration* (4 vols.), Hermann, Paris, 1952-63.

[D] J. Doob, *Probability in function space*, Bull. Amer. Math. Soc. **53** (1947), 17–30.

[H] P. R. Halmos, *Measure theory*, Van Nostrand, New York, 1950.

[K] A. Kolmogorov, *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Springer-Verlag, New York, 1933.

[M] A. deMoivre, *Approximatio ad summam terminorum binomii* $(a+b)^n$ *in seriem expansi* (original 1733), The Doctrine of Chances, 3d ed., Chelsea, New York, reprinted 1967.

[S-K] I. Segal and R. Kunze, *Integrals and operators*, McGraw-Hill, New York, 1968.

[W] N. Wiener, *Differential space*, J. Math. Phys. MIT **2** (1923), 131–174.

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Noncommutative Noetherian rings, by J. C. McConnell and J. C. Robson. Pure and Applied Mathematics. Wiley Interscience, John Wiley and Sons, New York, 1987, xiv+596 pp., $138.00. ISBN 0-471-91550-5

A ring R is said to be *right Noetherian* if every right ideal of R is finitely generated. There are two equivalent conditions: The *ascending chain condition* (every ascending chain of right ideals becomes stationary) and the *maximal condition* (every nonempty set of right ideals contains maximal elements). It has been 100 years since Hilbert [H] proved his basis theorem, which nowadays is stated in the following form: If R is a Noetherian ring, so is $R[x]$, the polynomial ring over R in one variable.

Hilbert used his result to conclude that certain rings of invariants were finitely generated. The name "Noetherian" honors