where either $\rho(x) \to \infty$ or $\rho(x) \to 0$ as $x \to \infty$. When such a factor $\rho(x)$ appears the system is said to be resonant, and otherwise nonresonant. The knowledge that resonance can occur goes back to Perron (1930) but the first systematic analysis of resonance and nonresonance appears to be due to Atkinson (1954).

Other applications of Levinson's theorem that the reviewer is familiar with, but not covered in this book, include spectral and scattering theory for ordinary and partial differential operators and to wave propagation in stratified fluids. The reviewer feels sure that there are other applications or possible applications to areas with which he is unfamiliar.

This book is written in the author's usual elegant style. The exposition is crisp, the explanations and proofs are clear. It can certainly be recommended for the bookshelf of anyone interested in its subject matter. Indeed, it can be recommended for anyone who enjoys reading well-written mathematics and learning a bit about a small, but important, corner of mathematics.

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*Conformal geometry*, Ravi S. Kulkarni and Ulrich Pinkall, eds. Aspects of Mathematics, Friedr. Vieweg & Sohn, Braunsweig and Wiesbaden, 1988, viii + 236 pp. ISBN 3-528-08982-2

The book under review is the proceedings of a seminar on conformal geometry held at the Max-Planck Institute in 1985–1986. This subfield of differential geometry is rather vast and multifaceted, and therefore it would be impossible for a single volume to deal with all aspects of this subject. The book contains several survey articles which deal with various aspects of conformal Riemannian structures, from the points of view of both topology/synthetic-geometry and local differential geometry. From both points of view flat conformal structures play a central role and all of the papers in this volume deal with at least some aspects of the theory of conformally flat manifolds. For this reason this review will concentrate on conformally flat manifolds and will attempt to put in perspective some more recent advances in this subject.

Perhaps the most familiar approach to a conformal structure on a manifold is a conformal equivalence class of Riemannian metrics: Two Riemannian metrics on a smooth manifold $M$ are conformally equivalent if they give rise to the same measurement of angles in the tangent spaces. Analytically this means that the metric tensors $g_1$, $g_2$ are scalar multiples of each other: For any two tangent vectors $X, Y \in T_xM$ the metric tensors are related by:

$$g_1(X, Y) = f(x)g_2(X, Y)$$

for a smooth function $f: M \to \mathbf{R}$. In general this represents quite a violent change in the behavior of geodesics although other aspects of the Riemannian geometry (for example, characteristic classes) behave more tamely under this change.

Conformal structures in dimension $> 2$ differ substantially from conformal structures in two dimensions, where they coincide with complex-analytic structures. Perhaps the most striking and basic result in this direction is Liouville's theorem that a conformal transformation between two connected subdomains of euclidean space (of dimension $> 2$) is the restriction of a global conformal diffeomorphism of the $n$-sphere, (i.e., a Möbius transformation of several variables). (Such Möbius transformations constitute the Lorentz orthogonal group $SO(n+1, 1)$ acting on a sphere embedded as a quadric hypersurface $S^n \subset \mathbf{RP}^{n+1}$.) Geometric structures locally modelled on this geometry are called *Möbius structures.*

Of paramount importance in the global study of conformal structures on manifolds are those which are locally conformally equivalent to euclidean space, i.e., those for which for each point $x \in M$ there is a conformal change of metric which is isometric to the flat metric on euclidean space. Gauss proved that every surface is conformally flat in this sense; however, in higher dimension (for example, dimension 3), it is much rarer to be conformally flat. Such higher-dimensional *flat conformal structures* turn out to be the same thing as Möbius structures. Their global properties were first investigated by Kuiper [K1, K2]. Some, but certainly not all, Möbius structures arise on quotients of domains in $S^n$ by discrete groups of conformal transformations [ST]. (Such structures are called "uniformizable.") A basic idea which launches this study is the process of "developing" the universal covering in the sphere: By "analytic continuation" of the conformal coordinate charts, one builds a conformal immersion of the universal covering $\widetilde{M}$ into $S^n$—it follows immediately that a closed simply connected conformally flat manifold is conformally equivalent (in particular homeomorphic) to the sphere. Indeed the Poincaré conjecture is equivalent to the conjecture in conformal geometry that every compact simply connected 3manifold admits a flat conformal structure. However, relatively simple examples of closed 3-manifolds (for example, torus bundles over the circle not covered by the 3-torus [G]) fail to admit flat conformal structures. On the other hand, Möbius inversion in spheres shows that a geometric ball is conformally equivalent to its exterior in the sphere—it follows that connected sums of conformally flat manifolds possess flat conformal structures [Kl]. Since all but three of the eight geometries, Thurston has conjectured uniformize 3-manifolds (compare [Sc]) can be made conformally flat, an enormous class of closed 3-manifolds admits flat conformal structures (see [GLT, Kv] for further examples).

In 1976, Thurston (unpublished) gave a description of Möbius structures on manifolds of real dimension 2 in terms of 3-dimensional hyperbolic geometry. The precise statement is the following: Let $S$ be a surface; then to every Möbius structure on $S$ there corresponds a hyperbolic structure and measured geodesic lamination on the corresponding hyperbolic surface $M$. Furthermore the resulting map

$$\Theta: \mathbf{C}P^1(S) \to \mathfrak{T}(S) \times \mathfrak{ML}(S)$$

is a diffeomorphism from the deformation space of Möbius structures to the product of Teichmüller space with the Thurston space of measured geodesic laminations. The geometric construction associates to the Möbius structure a "locally convex pleated map" from the universal covering of $S$ to hyperbolic space equivariant with respect to the holonomy representation of $\pi_1(S)$. One pictures $\widetilde{M}$ as a totally geodesic hyperbolic 2-manifold inside hyperbolic 3-space "bent" along a system of disjoint complete geodesics (the "bending lamination"). The first Thurston parameter (the hyperbolic structure) describes the intrinsic geometry of this developable surface and the second Thurston parameter (the bending measure) describes the extrinsic geometry (how the surface is bent). If the Möbius structure arises from a Kleinian group ($S$ is the quotient of a domain $\Omega$ by a discrete group $\Gamma$), then the pleated surface is the boundary of the convex hull of the complement of $\Omega$ in $\mathbf{H}^3$. There is a purely conformal description of this geometry, whereby the supporting hyperplanes to the pleated surface correspond to maximal geometric discs $D$ in $\widetilde{S}$ and the bending lamination is the boundary of the Poincaré convex hull of the closed set of "invisible" points on $\partial D \hookrightarrow \mathbf{H}^3$. Sullivan has proved the important result that if $\Omega$ is simply connected, then the natural map from $\Omega$ to the boundary of the convex hull of its complement is $k$-quasiconformal. Thus for "uniformizable structures" (those for which the developing map is injective), the point in Teichmüller space corresponding to the Thurston parametrization is a bounded distance from the point in Teichmüller space recording the conformal structure. (A careful and scholarly account of such convex hulls and the relation to geodesic laminations has been given by Epstein and Marden [EM]. A family of geometric structures interpolating between these two parameters in Teichmüller space has recently been constructed by Labourie [L].) Recently Kulkarni and Pinkall have worked out the higher-dimensional theory of flat conformal structures in terms of maximal geometric discs and bending lamina. In particular they have proved that the bending lamination has differentiability class $C^{1,1}$.

In [SY], Schoen and Yau considered the scalar curvature of a conformal Riemannian metric as an invariant of a flat conformal structure. Using the spectral characterization of amenability due to Brooks [B], they classified flat conformal structures on closed manifolds with amenable holonomy group (such a conformally flat manifold is covered by a sphere, in Hopf manifold, or a flat torus, extending similar results [K2, G, F]). Assuming a conformally flat Riemannian metric with positive scalar curvature, the developing map embeds the universal covering space diffeomorphically onto a domain in $S^n$ whose complement has Hausdorff dimension $< n/2$. In particular, the holonomy representation embeds the fundamental group of $M$ as a "Kleinian group," a discrete group of conformal transformations acting properly on a subdomain of the sphere. Indeed by "stabilization," their work indicates a close connection between the geometric condition of injective development (being a quotient of a domain) and the analytic condition of positive scalar curvature: If $M = (S^n - \Lambda)/\Gamma$ is a closed flat conformal manifold with injective development, for any $k > 0$, $\Gamma$ acts properly discontinuously and freely on $S^{n+k} - \Lambda$ and the quotient $(S^{n+k} - \Lambda)/\Gamma$ is a closed conformally flat manifold of dimension $n + k$. By taking $k$ sufficiently large, this manifold will have positive scalar curvature [SY, Theorem 4.7], providing a kind of converse to the classification result above.

In another direction, new examples of flat conformal structures on 3-manifolds have been discovered (independently) by Gromov, Lawson, and Thurston [GLT] and Kapovich [Kv] (see also [K3]). Uniformizable flat conformal structures exist on twisted $S^1$-bundles over closed hyperbolic Riemann surfaces (twisted $S^1$-bundles over a torus do not support flat conformal structures). These structures bound hyperbolic structures on the corresponding 2-disc bundles over surfaces. However, it is very difficult to determine precisely which circle bundles admit flat conformal structures. Kapovich [Kv] has found such manifolds $M$ for which the space of flat conformal structures on $M$ has more than one component and Apanosov [A2] has examples where the space of uniformizable flat conformal structures is disconnected.

Conformally flat manifolds can be used to construct examples of compact complex manifolds with free nonabelian fundamental group. (The fundamental group of a nonsimply-connected compact Kähler manifold is never free.) The *twistor construction* associates to a conformally flat 4-manifold $M$ an $S^2$-bundle $E$ over $M$ which is a complex manifold (with a holomorphic flat projective structure). (The fibers of $E_x$ over $x \in M$ consist of the complex structures on the tangent space $T_xM$.) Applying this construction to flat conformal structures on connected sums of $S^1 \times S^3$, one obtains interesting compact complex manifolds with free fundamental group. (This was observed by N. Hitchin; see [N].)

Unfortunately, mounting evidence suggests that extending the marvelous theory of quasiconformal deformations of Kleinian groups (developed by Grotszch, Teichmüller, Ahlfors, Bers, etc.) to higher dimensions is unlikely. One construction for flat conformal structures in higher dimensions is the following. Given a compact hyperbolic manifold $M$ containing a closed totally geodesic hypersurface $\Sigma$, there is a one-parameter family of flat conformal structures on $M$ obtained by splitting $M$ along $\Sigma$ and inserting a "crescent" (diffeomorphic to $\Sigma \times [-\varepsilon, \varepsilon]$) in its place. This "bending" construction (so called since it generalizes Thurston's construction mentioned above—compare the "Mickey Mouse" example in [T, §8.7.3]). This construction has been discovered independently by various authors in different contexts: Maskit, Apansov [A], Thurston, LaFontaine, Kouroniotis [Ko], Johnson and Millson [JM], and perhaps others. These particular deformations are unusually explicit and computable: Using them, Johnson and Millson [JM] have proved that the deformation space of flat conformal structures (locally a real algebraic variety) is typically not a smooth manifold. This already indicates difficulties in trying to extend the Ahlfors-Bers theory to higher dimensions: The presence of nontrivial integrability conditions (conveniently absent in the classical case) forces singularities in the deformation space.

Another difference between the classical case (two-dimensional conformal geometry) and higher dimensions is Ahlfors' finiteness theorem: For a finitely generated group $\Gamma$ of conformal transformations whose domain of discontinuity is $\Omega \subset S^2$, the corresponding Riemann surface $\Omega/\Gamma$ has finite type. Kapovich and Potyagailo [KP] have shown that topological considerations in dimension 3 preclude even the weakest extension of this phenomenon to higher dimension: Finitely generated groups $\Gamma$ acting conformally on $S^3$ exist, for which the corresponding uniformizable conformally flat manifold $\Omega/\Gamma$ does not even have finitely generated fundamental group. (In fact, $\Gamma$ cannot even be finitely presented, further contrasting the two-dimensional (classical) case.)

Thus two analytic cornerstones of the theory of classical Kleinian groups (the measurable Riemann mapping theorem and the Ahlfors finiteness theorem)—called by Sullivan [Sul] "almost the axioms for a good theory of Kleinian groups"—must undergo substantial modifications in a higher-dimensional theory.

The first article in the book under review, by Kulkarni, deals with Möbius structures, providing a useful introduction to this subject, with some of its historical origins. In particular Liouville's theorem is proved. The second paper in this volume (also by Kulkarni) deals with the classification of conformal transformations (and connected groups thereof) up to conjugacy. The main result of this paper is the following interesting criterion for a Zariski dense group $\Gamma$ of conformal transformations of an odd-dimensional sphere $S^{2n-1}$ to be discrete: If $\Gamma$ contains no elliptic elements (an elliptic element is one which fixes a point is hyperbolic space, or equivalently one represented by a semisimple with all eigenvalues of norm 1), then $\Gamma$ must be discrete. This is a general property shared by subgroups of real semisimple Lie groups which possess a compact Cartan subgroup (such as automorphism group of Hermitian symmetric spaces and period domains). In that case there is an open set of elliptic elements and if $\Gamma$ is Zariski dense, it is either dense or discrete (normalizing the Lie algebra of the identity component of its closure is an algebraic condition)—and the former is excluded since $\Gamma$ is disjoint from the open set of elliptic elements.

A reference to a "fairly standard lemma on linear groups" (page 61, Lemma 6.2) is not supplied; this lemma states that a linear representation of a finitely generated group $G$, all of whose eigenvalues are of norm 1, is an iterated extension of orthogonal representations. The argument is briefly as follows (see [CG]). Let $V$ be a $G$-module with this property. By the Jordan-Hölder theorem, it suffices to assume that $V$ is irreducible and deduce that the entries of the matrices representing elements of $G$ are uniformly bounded—then $G$ will be contained in a compact group and conjugate to a subgroup of the orthogonal group. By the Burnside lemma, irreducibility implies there exists a finite set $g_1, \dots, g_l \in G$ and complex scalars $c_1, \dots, c_l$ such that each elementary matrix $e_{ij}$ decomposes as a linear combination:

$$e_{ij} = \sum_{k=1}^{l} c_{ijk} g_k.$$

Since each element of $G$ has eigenvalues of norm 1, the absolute value of each trace is bounded by $\dim(V)$. Choosing $C > c_{ijk}$, the $ij$ th entry of the matrix corresponding to an arbitrary $\gamma \in G$ is now uniformly bounded:

$$|\gamma_{ij}| = |\mathrm{trace}(\gamma e_{ij})| \leq C \dim(V).$$

I think this useful trick deserves to be better known.

More analytic aspects of conformal differential geometry are represented by the papers by Lafontaine, Kühnel, Rickmann, Rademacher, and Pinkall. Lafontaine's first paper in this book, entitled "Conformal geometry from the Riemannian viewpoint," treats the conformal geometry of a Riemannian manifold from the point of view of tensor analysis. This paper gives a self-contained account of the theory of the Weyl and Schouten conformal curvature tensors and is one of the best expositions of this material I have seen. The general theory is applied to conformally flat hypersurfaces in $\mathbb{R}^n$, the case $n = 4$ requiring special consideration since conformal flatness is detected by a higher-order condition. Lafontaine's second paper gives a proof of the theorem of Obata and Lelong-Ferrand that a compact manifold with a noncompact group of conformal automorphisms is conformally equivalent to a sphere. Kühnel's paper discusses conformal mapping between Einstein manifolds. A prominent role is played by "concircular mappings" (conformal mappings preserving geodesic circles). Apparently the literature is full of contradictions throughout and this paper achieves the admirable goal of setting the record straight on presenting counterexamples to the various errors in the literature. An appendix is devoted to the proofs of two "standard wrong theorems" in this subject.

Rickmann's paper surveys quasiregular mappings (the generalization of quasiconformal mappings to noninjective mappings) and details some of the interesting recent developments in this subject. Rademacher's paper deals with immersions of conformally flat manifolds and gives various obstructions to the existence of such immersions of some of the basic examples of conformally flat manifolds (for example, connected sums, products of hyperbolic manifolds with the circle, etc.). The methods use the Schouten and Weyl tensors and this paper fits in nicely with some of the other papers in this volume. Pinkall's article considers which conformally flat manifolds admit conformal immersions as hypersurfaces in $\mathbb{R}^n$. A rather striking result is that for $n \geq 4$, such a manifold must be a "classical Schottky manifold," a connected sum of Hopf manifolds along totally umbilic spheres. It is also shown that such flat conformal structures are rather special in that not all flat conformal structures on these manifolds are classical Schottky.

In general, this is an interesting book with a diverse set of techniques and results. It should make interesting reading for anyone who wants to whet his or her appetite in this fascinating area of geometry.

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*Real analysis and probability*, by R. M. Dudley. Wadsworth & Brooks/Cole Math Series, $52.95. ISBN 0-534-10050-3

There exist many books on each of the areas of real analysis and probability, including some which attempt to treat both subjects in the same treatise. Therefore, one may ask for a compelling reason to publish yet another work on this conjunction of well-established subjects.

Real analysis at the graduate level traditionally consists of measure and integration theory with an introduction to functional analysis. The prevailing tendency has been to treat these topics at an abstract level, with little or no historical commentary and almost no explicit reference to either the motivation or the applications of the material. At the same time we are told that measure theory provides a rigorous foundation for probability theory, while functional analysis has its origins in the theory of integral equations and is central to the modern theory of partial differential equations, among other things. For some students these connections might bring the subject more to life, but traditional approaches have opted for the path of efficient pedagogy, leaving the student to fill in the gaps for himself or herself.

In the case of probability theory, the development of measure and integration theory is long overdue. The earliest form of the weak law of large numbers was proved by Jakob Bernoulli [Be] in 1713; the first version of the central limit theorem was published by Abraham de Moivre [M] (at the age of 66) in 1733, exactly 200 years prior to the measure-theoretic framework which Andre Kolmogorov [K] introduced in 1933. Perhaps the first person to have