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Initial-boundary value problems and the Navier–Stokes equations, by Heinz-Otto Kreiss and Jens Lorenz. Academic Press, New York, 1989, 398 pp., $54.50. ISBN 0-12-426125-6

When Einstein died and went to heaven, the story goes, he asked the Maker about the Grand Unified Theory. The Great Equationist unveiled a beautiful theory and soon the celestial seminar room was filled with harmonious formulae. Encouraged by such a divine answer, Einstein then asked about the Theory of Turbulence. The seminar room suddenly became dark and a thunderous voice ordered the impudent soul to leave instantly.

Allowing for poetic license at the microscopic level, the Navier–Stokes equations are nothing but Newton's laws for fluid and gases. As such, they apply universally. Neglecting thermodynamical effects, the fluid or gas is described by five functions—the three-component velocity vector field, the pressure and the density. Conservation of mass and momentum provide four equations for the unknowns. The fifth equation distinguishes between the compressible and incompressible cases. In the compressible case variations in the density induce variations in the pressure. In this case an equation of state $p = f(\rho)$ is postulated, giving the pressure as a function of the density. The function f is assumed to be known from thermodynamical considerations. Incompressibility, on the other hand, is the property that the density is constant along particle trajectories. In view of the continuity equation (conservation of mass) incompressibility is expressed in the relation $\operatorname{div} u = 0$ which is the fifth equation in this case (u is the velocity). The resulting equations are the compressible (resp., incompressible) Navier–Stokes equations if viscous effects are retained or compressible (resp., incompressible) Euler equations if these effects are neglected.

There are three fundamental questions concerning these equations. First, do solutions exist? Secondly, how do solutions depend on the physical parameters? Thirdly, what is the relevant long-time behavior of solutions?

First, the question of existence of solutions. This question has two aspects: one regarding well-posedness and the other regarding global existence. Well-posedness of an initial value problem requires the local existence, uniqueness and continuous dependence on initial data of solutions. This is relatively well understood in classes of smooth functions but can become a very serious problem in classes of functions which are not smooth. There are strong mathematical and physical reasons to study nonsmooth initial data. These reasons are obvious in the hyperbolic cases, because solutions develop shocks. In the incompressible case, important structures such as vortex patches, vortex sheets or density stratified fluids are not smooth. The mathematical study of vortex patches was initiated by Yudovitch [Y]. He proved the global existence of weak solutions of the two-dimensional incompressible Euler equations for initial velocities whose curl (vorticity) is in L^∞ . If the initial vorticity is in L^p , $1 < p < \infty$ then existence is known, but uniqueness is not. If the initial vorticity is a measure supported on a smooth curve (vortex sheet) then the whole concept of solution may have to be revised, perhaps to include measure-valued solutions [MdP]. The corresponding problem for the incompressible Navier–Stokes equations is better behaved [GM].

Whether smooth solutions to the incompressible Navier–Stokes and Euler equations in three space dimensions exist for all times or not is a central question which remains widely open. In the case of the Navier–Stokes equations, the fundamental result of Leray ([L], 1934) established the global existence of weak solutions. The uniqueness of these solutions has not yet been either proved nor disproved; there are known sufficient conditions which guarantee uniqueness [S]. The regularity properties of these solutions are not sufficiently understood. It is known that the spatial BV norm is bounded uniformly in time [C2, CLMS] and various temporal averages of higher derivatives are also a priori bounded [C2, FGT]. Also, the Hausdorff dimension of the set of possible space-time singularities of any suitable weak solution is at most one [CKN]. Many of these results are described in existing monographs on this subject [T, CF, vW].

The second fundamental question is that of dependence on physical parameters. Typically, problems in this class involve singular limits. Here are two of the most important questions of this nature. Are the incompressible equations the limit of the slightly compressible equations? Are the Euler equations the inviscid limit of the Navier–Stokes equations? The first question is known to have a positive answer [KM] in the case of smooth solutions. Regarding the second question in the absence of boundaries the answer is yes, for short times [K] or at least as long as the Euler solution is smooth [C1]. In the presence of boundaries the answer is in general no, because of discrepancies in the boundary behavior (boundary layers). The central problem of describing the inviscid limit is open in this case. The incompressible limit is discussed in the book [M], the inviscid limit in the book [CF].

The third fundamental question is that of the relevant long-time behavior of solutions. This has again two aspects, deterministic and statistical. To appreciate the difficulty of this type of problem, it is enough to recall the complexity encountered in the study of three or more nonlinear autonomous ordinary differential equations. Solutions which start very close to each other can behave in very different ways; the system is still deterministic but only to the Infinite Precision Eye. To make things worse (or more interesting?) such systems in general display a mixture of apparently random (unpredictable) and coherent (predictable) behavior. The complexity encountered in dynamical systems is not an artifact: real physical systems display it. For the incompressible Navier–Stokes equations, the long-time behavior is finite dimensional in the sense that there exists a global (universal) attractor of finite Hausdorff and fractal dimensions [FT]. The finite dimensionality of the attractor establishes a profound connection between the mathematical theory of the Navier–Stokes equations and conventional turbulence theory [LL, R, CFMT]. In a sense the Navier–Stokes equations are a finite-dimensional dynamical system but whether there exists a smooth finite-dimensional manifold which describes all the long-time behavior (inertial manifold) is still unknown.

The mathematical study of statistical solutions of the Navier–Stokes equations was initiated by E. Hopf [H]; the fundamental existence theorem is due to Foias [F]. Statistical solutions are probability measures in function space describing the solution of the Navier–Stokes equation with random initial velocities. These topics as well as existence of homogeneous statistical solutions and much of the known mathematical results on statistical solutions are described in the monograph [VF].

Most of the book under review (Chapters 2, 3, 5–8) is devoted to the study of the Cauchy problem (initial value problem) with smooth data. The energy method and the Laplace transform method are the main themes; they are clearly and accessibly illustrated in a variety of situations. Several definitions of well-posedness are discussed in detail. The presentation follows a natural order of increasing difficulty: from constant-coefficient Cauchy problems (Chapter 2) to linear variable coefficient problems in one dimension (Chapter 3), nonlinear systems in one dimension (Chapter 5), then linear systems in several dimensions (Chapter 6). Various typical boundary conditions in one (Chapter 7) and several (Chapter 8) dimensions for hyperbolic, parabolic and mixed hyperbolic-parabolic systems are described. Kreiss made important contributions to the general theory of hyperbolic Cauchy and initial-boundary value problems; while never becoming too technical, the book provides deep insight into these matters. Chapter 4 is devoted to the Burgers equation while only Chapters 9 and 10 concern the incompressible Navier–Stokes equations. The book is very readable and should prove to be a useful introduction to important aspects of this vast subject.

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PETER CONSTANTIN
UNIVERSITY OF CHICAGO

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Mathematical problems from combustion theory, by Jerrold Bebernes and David Eberly. Applied Mathematical Sciences, vol. 83, Springer-Verlag, New York, 1989, 177 pp., $34.00. ISBN 0-387-97104-1

Combustion involves the liberation of energy by chemical reaction. Typically, the rate of energy release is a strongly sensitive function of temperature. Doubling the temperature, for example, may well increase the rate by a factor of ten thousand. Combustible materials, essentially inert at room temperature, can therefore ignite rapidly and explode when sufficiently heated.