H-infinity control. Nevertheless the book is a good introduction into an active area of research and the reader who is willing to invest some time and effort should be well rewarded.

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Operators and representation theory: Canonical models for algebras of operators arising in quantum mechanics, by P. E. T. Jorgensen. North-Holland Mathematics Studies, vol. 147, North-Holland, Amsterdam, 1988, viii+338 pp., Dfl. 175.00. ISBN 0-444-70321-7

The subtitle of the book under review is *Canonical models for algebras of operators arising in quantum mechanics*. In the preface, the author states that he has "picked certain subjects from the theory of operator algebras, and from representation theory, and showed that they may be developed starting with Lie algebras, extensions, and projective representations." The distinctive point of view arises from the consideration of algebras of *unbounded* operators, which fall outside the usual theory of C^* algebras and von Neumann algebras. Analytic properties of these operators, such as essential self-adjointness, are treated using spaces of C^∞ and analytic vectors for an appropriate Lie group action.

The basic operator of interest here is the Hamiltonian (total energy) operator H of a quantum-mechanical system. The problem is to determine the spectrum and (generalized) eigenvector decomposition of this self-adjoint operator. In Sophus Lie's creation of the theory of continuous groups in the 1870s, the example of Galois theory served as a model: an algebraic equation has its group (the symmetry group of the roots of the equation), and the solutions of the equation can be studied in terms of the structure of the associated group. Lie-theoretic approaches to quantum-mechanical problems follow a similar pattern: given the operator H in some concrete realization (such as the Schrödinger picture), we look for a Lie algebra $\mathbf{g}$ and group G that are naturally associated with H.

Typically this means that there is a unitary representation of G on the Hilbert space of the system, such that H can be identified with the action of an element in the universal enveloping algebra $U(\mathbf{g})$. The structure of the unitary representations of G and their decomposition into irreducibles then yields information about the spectrum and eigenfunctions of H.

Implicit in this program is the assumption that the unitary representation theory of G is sufficiently developed so that it provides a viable noncommutative Fourier analysis (the groups G that occur are invariably nonabelian, because of the noncommuting nature of position and momentum variables in quantum mechanics). Fortunately this is possible for large classes of connected finite-dimensional Lie groups, thanks to the monumental contributions of E. Cartan, H. Weyl, I. M. Gelfand, I. E. Segal, Harish-Chandra, G. Mackey, A. Kirillov, and many other workers in representation theory in this century.

Fundamental tools in this case are the use of (quasi-) invariant measures on homogeneous spaces for G to construct *induced representations*, Mackey's *imprimitivity theorem* characterizing such representations, and the Kirillov correspondence between representations and coadjoint orbits. For studying representations by Lie algebraic methods, one introduces the space of *smooth vectors* for a representation, which is a module for the associative algebra $U(\mathbf{g})$. Jorgensen briefly reviews these matters, emphasizing the relation between unitary representations of G and self-adjoint representations of $\mathbf{g}$. The recent book of M. Taylor [4] covers some of the same ground in more detail, and gives an excellent introduction to representation theory from the point of view of applications to analysis and geometry.

In Jorgensen's book the representation theory of connected nilpotent Lie groups is applied to the example of a Hamiltonian for a particle in a curved magnetic field, and to scalar Schrödinger operators with polynomial potential. Finding the group G is quite easy in these cases: the operator H is assumed to be of the form $X_1^2 + \cdots + X_r^2$, where X_i is a first-order differential operator with real polynomial coefficients. The Lie algebra $\mathbf{g}$ generated by $X_1, \ldots, X_r$ is finite-dimensional and nilpotent. Furthermore, H comes from a hypoelliptic operator in $U(\mathbf{g})$, by a celebrated theorem of L. Hörmander. The representation-theoretic machinery just described can then be applied to determine the spectral properties of H, as Jorgensen shows in Chapters 6 and 7.

In the algebraic approach to quantum mechanics, pioneered by I. E. Segal, the observable quantities of physical system comprise a C^* algebra $\mathscr{A}$, and each state of the system gives rise to a *-representation of $\mathscr{A}$ on a Hilbert space. A symmetry group G (or Lie algebra $\mathbf{g}$) for the system acts by *-automorphisms (resp., derivations) on $\mathscr{A}$ (resp., a dense subalgebra of $\mathscr{A}$). When this action can be implemented in a representation of $\mathscr{A}$, one obtains a projective representation of G (resp., $\mathbf{g}$). In the last third of Jorgensen's book several examples of such models are examined. One issue is the exponentiation of the action of $\mathbf{g}$ when $\mathbf{g}$ is infinite-dimensional (e.g. the Lie algebra of smooth vector fields on the circle, which occurs in conformal field theories, cf. [2, 3]). Another is the classification of smooth Lie group actions on a "noncommutative torus" (motivated by A. Connes theory of noncommutative differential geometry).

The book concludes with an appendix on integrability of Lie algebra representations, and forty-five pages of references. Curiously, about two-thirds of the articles listed are never cited in the text (e.g. the nineteen papers of Harish-Chandra).

Unfortunately, Jorgensen's book is marred by quite a few logical blunders and errors of terminology. Despite the emphasis on the role of Lie algebras and their extensions throughout the book, there is a persistent confusion between quotient algebras, subalgebras, and linear subspaces of a Lie algebra. This begins already on page 5, with the definition of *central extension* of a Lie algebra (the algebra being extended incorrectly appears as a *subalgebra* rather than a *quotient*). Another example of this sort of mistake occurs on page 125, in connection with Lie algebras for a magnetic field Hamiltonian, and in the use of coexponential coordinates. The construction of the cocycle representation of a pair Γ, $\widehat{\Gamma}$, where Γ is a discrete Abelian group, on page 176ff. involves a circular argument. On page 215 the Lie algebra $\mathbf{u}(n, 1)$ is described as "an extension of $\mathbf{u}(n)$ by the Abelian Lie algebra $\mathbf{C}^n$ ", where in fact $\mathbf{p} = \mathbf{C}^n$ is the subspace (not subalgebra) for the Cartan decomposition $\mathbf{g} = \mathbf{k} + \mathbf{p}$.

A more serious analytic error occurs in the discussion of analytic domination and integrability of Lie algebras of operators. Here the group G is $\mathrm{SL}(2, \mathbf{R})$ with maximal compact subgroup $K = \mathrm{SO}(2)$. Estimate (5.5.9), stated (without proof) on page 59, is used to conclude on page 68 that the K-finite vectors for a unitary representation of G are *entire*. If true, this would imply that the matrix entry functions of such vectors extend holomorphically to the complex group $\mathrm{SL}(2, \mathbf{C})$. But this is easily seen to be impossible, e.g. by considering the spherical functions for G (cf. [1, §8] for general nonextendability results for unitary representations of noncompact semisimple groups). This error does not contradict the integrability theorem Jorgensen is aiming at, since local analytic extendability suffices. However, it undermines his exposition of these results.

In summary, Jorgensen's book, despite its technical flaws, presents an overview of an interesting range of recent developments in operator and representation theory.

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