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SUBCRITICALITY, POSITIVITY, AND GAUGEABILITY OF THE SCHRÖDINGER OPERATOR

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1. Introduction

We investigate properties of the Schrödinger operator $H := -(\Delta/2) + V \geq 0$ in $R^d (d \geq 3)$ in the following three aspects:

(I) Subcriticality: Intuitively, the idea is that if $H \geq 0$ is subcritical, then it should be possible to perturb H by small perturbations and still keep its nonnegativity. More precisely, we have the following assertions:

(a) For any $q \in B_c$ (B_c denotes the class of bounded Borel functions with compact support), there exists an $\varepsilon > 0$ such that $-(\Delta/2) + V + \varepsilon q \geq 0$.

(b) There exists a function $q \in B_c$, $q \leq 0$ and $q \not\equiv 0$ a.e. such that $-(\Delta/2) + V + q \geq 0$.

There have been two other definitions of subcriticality:

(c) (B. Simon [7]) There exists $\beta > 0$ such that $-(\Delta/2) + (1 + \beta)V \geq 0$.

(d) (M. Murata [6]) There exists a positive Green function $G^H(\cdot, \cdot)$ for H.

(II) Strong Positivity:

(e) There exists a positive solution $u > 0$ of $Hu = 0$ with the limit: $\lim_{|x| \to \infty} u(x) > 0$.

(f) There exists a solution u of $Hu = 0$ with $c' \geq u \geq c > 0$.

(g) There exists a solution u of $Hu = 0$ with $u \geq c > 0$.

(III) Gaugeability: Let $\{X_t : t \geq 0\}$ be the Brownian motion in R^d and let E^x denote the expectation over the Brownian paths starting from $x \in R^d$. Put $u_0(x) := E^x[\exp(-\int_0^\infty V(Xs)\,ds)]$.

(h) $u_0(x) \not\equiv \infty$ in R^d.

(i) $u_0(x)$ is bounded in R^d.

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For any y in R^d, we define the y-conditional Brownian motion of Doob type (see [10]) and use E_y^x to denote the expectation over the y-conditional Brownian paths starting from x. Put

$$u_0(x, y) := E_y^x \left[\exp\left(-\int_0^\xi V(Xs)\,ds\right)\right], \qquad x, y \in R^d,$$

where ξ is the lifetime of the process.

(j) $u_0(x, y) < \infty$ for some (x, y) in $R^d \times R^d$, $x \neq y$.

(k) $u_0(x, y)$ is bounded in $R^d \times R^d$.

Our main result is the equivalence of all the assertions (a) through (k) listed above for a large class of potentials V given below.

2. Restricted Kato class K_d^∞

For a function V in K_d^{loc}, $d \geq 3$ (see [8] for definition of the Kato classes K_d^{loc} and K_d), we add a similar Kato condition around the point at ∞ and then form a new class K_d^∞ called the restricted Kato class:

$$(1) \quad K_d^\infty := \left\{ V \in K_d^{\text{loc}}\colon \lim_{A\to\infty} \left[\sup_{|x|\geq A} \int_{|y|\geq A} \frac{|V(y)|}{|y-x|^{d-2}}\,dy\right] = 0 \right\}.$$

It is easy to see that $K_d \cap L^1(R^d) \subseteq K_d^\infty \subseteq K_d$. It can be verified by Hölder's inequality that K_d^∞ also contains the class of "short range potentials":

$$(2) \qquad \{V \in K_d\colon V(x) = O(|x|^{-\rho}) \text{ as } |x| \to \infty,\ \rho > 2\}.$$

We note that Murata [5] proved some part of the above-mentioned equivalences for subcriticality for potentials satisfying the condition in (2) with $\rho > 4$.

For $V \in K_d^\infty$, put $|||\, V \,||| := \sup_{x\in R^d} \int_{R^d} (|V(y)|/|x-y|^{d-2})\,dy < \infty$. We add two more assertions to the list in (I):

(l) There exists an $\varepsilon > 0$ such that for any $q \in K_d^\infty$ with $|||\, q \,||| < \varepsilon$, $-(\Delta/2) + V + q \geq 0$.

(m) There exists a function $q \in K_d^\infty$, $q \leq 0$ and $q \not\equiv 0$ a.e. such that $-(\Delta/2) + V + q \geq 0$.

3. MAIN THEOREM AND SKETCH OF THE PROOF

Theorem. *For any* $V \in K_d^\infty (d \geq 3)$, *the conditions* (a) through (m) *are equivalent.*

Sketch of the proof. Since $V \in K_d^\infty$, there exists a $r > 0$ such that

$$\text{(3)} \qquad \sup_{|x| \geq r} \left[C_d \int_{|y| \geq r} \frac{|V(y)|}{|x-y|^{d-2}} dy \right] < \frac{1}{2},$$

where $C_d = \Gamma((d/2)-1)/2\pi^{d/2}$. Let $D = \{x \in R^d : |x| > r\}$ and $B = \{x \in R^d : |x| < 2r\}$. Put $T := \tau_B + \tau_D \circ \theta_{\tau_B}$ (the shuttle time), where τ_U is the exit time from a domain U and θ is the shift operator on paths. We define the shuttle operator S_V in the Banach space $C(\partial D)$: for $f \in C(\partial D)$,

$$\text{(4)} \; S_V f(x) := E^x \left[T < \infty ; \exp \left(- \int_0^T V(Xs)\, ds \right) f(X(T)) \right], \qquad x \in \partial D.$$

By Khasmin′skii's lemma together with (3) and the arguments similar to those in [10], we can prove S_V is an integral operator with continuous kernel:

$$S_V f(x) = \int_{\partial D} \Phi(x, y) f(y) \sigma(dy) \quad (\sigma \text{ is the area measure}),$$

where

(5)

$$\Phi(x, y) = 9(d-2)^2 C_d^2 r^2 \times \int_{\partial D} \frac{E_z^x [\exp(-\int_0^{\tau_B} V(Xs)\, ds)]\, E_y^z [\exp(-\int_0^{\tau_D} V(Xs)\, ds)]}{|x-z|^d |y-z|^d} \sigma(dz), \qquad (x, y) \in \partial D \times \partial D.$$

Put $\lambda(V) := \lim_{n\to\infty} \sqrt[n]{\|(S_V)^n\|}$.

Introducing the shuttle operator S_V and its spectral radius $\lambda(V)$ is the key idea in connecting the seemingly different assertions in the list (a) through (m). In fact, we add a new equivalent assertion as a linkage among the assertions (a) through (m):

(n) $\lambda(V) < 1$.

$\lambda(V)$, as a function of V, has the following properties:

Lemma. (L1) *If* $|||\, V_n - V \,||| \to 0$, *then* $\lambda(V_n) \to \lambda(V)$.
(L2) *If* $V_1 \le V_2$ *and* $V_1 \not\equiv V_2$ *a.e., then* $\lambda(V_1) > \lambda(V_2)$.

Both properties are based on the integral kernel representation (5) in terms of path integrals. We also need a characterization of nonnegativity of H, which can be regarded as a higher dimensional version of a result by Chung and Varadhan [2].

Proposition A. *For* $V \in K_d^{\infty}$, $-(\Delta/2) + V \ge 0$ *if and only if* $\lambda(V) \le 1$.

We now sketch the proof of some nontrivial implications in connection with (n). (n) $\Leftrightarrow$ (h): This equivalence is mainly given by the equality:

$$(6)\quad E^x\left[\exp\left(-\int_0^{\infty} V(Xs)\,ds\right)\right] = \sum_{n=0}^{\infty}(S_V)^n g(x), \qquad x \in \partial D,$$

where $g(x) := E^x[T = \infty; \exp(-\int_0^T V(Xs)\,ds)]$. The idea behind the equality (6) is that almost every Brownian path in $R^d (d \ge 3)$ will shuttle finitely many times between ∂B and ∂D before it goes off to ∞.

(n) $\Rightarrow$ (l): Suppose $\lambda(V) < 1$. By (L1), if $|||\, q \,|||$ is small enough, then $\lambda(V+q) < 1$. Therefore $-(\Delta/2) + V + q \ge 0$ by Proposition A.

(m) $\Rightarrow$ (n): By (L2) and Proposition A, we have $\lambda(V) < \lambda(V+q) \le 1$.

(c) $\Rightarrow$ (n): For each $0 \le t \le 1+\beta$, put $f(t) := \ln[\lambda(tV)] = \lim_{n\to\infty}(1/n)\ln\|(S_{tV})^n\|$.

Since for each n, $\ln\|(S_{tV})^n\|$ is a convex function of t by using the stopped path integral and the Cauchy-Schwarz inequality, so is the limit $f(t)$. Since $f(t) \le 0$ in $[0, 1+\beta]$ by Proposition A and $f(0) < 0$ by the transient property of the Brownian motion in $R^d (d \ge 3)$, we obtain $f(1) < 0$, i.e. $\lambda(V) < 1$.

Another key idea is the connection between the Green function $G^H(x, y)$ and the conditional Feynman–Kac gauge (see Zhao [10]):

$$G^H(x, y) = G^{\Delta/2}(x, y) E_y^x\left[\exp\left(-\int_0^{\xi} V(Xs)\,ds\right)\right].$$

The proof of equivalences in the list (III) involves gauge and conditional gauge arguments similar to those in [1], [3] and [9].

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