L FUNCTIONS FOR THE GROUP G_2

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INTRODUCTION

The method of L functions is one of the major methods for analyzing automorphic forms. For example, the Hecke Converse Theorem gives an equivalence via the Mellin transform between holomorphic modular forms on the upper half plane and certain L functions associated to Dirichlet series, which have analytic continuation and functional equation. The classical theory of automorphic forms on the group GL_2 can be reinterpreted in terms of the spectral analysis of functions on the space $GL_2(k)\backslash GL_2(\mathbf{A})$ where $\mathbf{A} = \mathbf{A}_k$ is the adele ring of the number field k, ([W-1]). Using adelic language, Weil and Jacquet–Langlands have developed the Hecke Converse Theorem for GL_2 from a representation theoretic point of view ([W-2, J-L]). This leads to the problem of analyzing the class of automorphic representations of a reductive group, G. That is, we consider a $G(\mathbf{A})$ irreducible representation Π embedded in a suitable subspace of $G(k)\backslash G(\mathbf{A})$. By the general theory developed by Langlands ([L]) one can associate to Π a whole class of L functions parametrized by the finite-dimensional modules of an associated L group. The first two major questions that arise are whether these automorphic L functions have analytic continuation and functional equation. These questions are particularly relevant in that such automorphic L functions are closely tied to applications in number theory. For instance, it is expected from Langlands' philosophy that the L functions describing the arithmetic structure of certain algebraic varieties can be described in terms of automorphic L functions.

The problem of determining the analytic continuation and functional equation of general L functions associated to automorphic representations of reductive Lie groups and representations of the

L groups has been pursued by two different techniques. The first is the interpretation of the L function as arising in the constant term of Eisenstein series (the Langlands–Shahidi method). The second is the generalized Rankin–Selberg integral representation of the L function. We refer the reader to [GS] for an excellent description of both methods and a comparison of the various techniques used.

The second method was originally developed to study the analytic properties of the L function associated to the tensor product of automorphic representations of $GL_2(\mathbf{A})$. The nonvanishing of such Rankin–Selberg L functions on the line $\operatorname{Re}(s) = 1$ implies certain estimates on the Hecke eigenvalues of the local components of an automorphic cuspidal representation of $GL_2(\mathbf{A})$. In concrete terms this determines the location of an unramified local component of a global automorphic representation in terms of how *close* to the end of the complementary series such a representation must be.

The next step in the development of the Rankin–Selberg L functions is the work (J-PS-Sh). Namely here the tensor product of two automorphic representations of GL_n and GL_m is considered. The main use of such work is twofold. First, one can prove a Strong Multiplicity One Theorem for cusp forms on $GL_n(\mathbf{A})$. This means simply that two global cuspidal $GL_n(\mathbf{A})$ representations $\Pi = \otimes_v \Pi_v$ and $\sigma = \otimes_v \sigma_v$ (with the same central character) are equal if and only if $\Pi_v \cong \sigma_v$ for almost all v. Second, the Rankin–Selberg L function is used to determine the predicted classification of automorphic representations of $GL_n(\mathbf{A})$. That is, each automorphic representation is given as a subquotient of an induced module with cuspidal data, where such data is determined uniquely up to associate class [J-S-1,2].

Subsequently in [G-PS-R] a theory for Rankin products is developed for groups of the form $GL_n \times G'$, G' a classical group. The specific example of the triple tensor product of GL_2 automorphic representations is given in [PS-R].

The common feature of the above examples is a setup of the following form. Namely we are given three reductive groups G_1, G, and H so that $G \supset G_1$ and $H \supset G_1$. Then we consider the corresponding adelic groups $G(\mathbf{A})$, $G_1(\mathbf{A})$, and $H(\mathbf{A})$. We let $f_{\Pi_1} \in \Pi_1$, a cuspidal irreducible representation of $G(\mathbf{A})$. If P_H is a maximal parabolic subgroup of H, then $P_H = M_H \cdot U_H$ where M_H is a Levi subgroup and U_H is the unipotent radical

of P_H. We let δ_{P_H} be the corresponding modular function associated to the adjoint action of P_H on its Lie algebra. Then δ_{P_H} is a character on P_H. We let Π_2 be a cuspidal representation of $M_H(\mathbf{A})$. Then with this data we form the induced module $\text{ind}_{P_H(\mathbf{A})}^{H(\mathbf{A})}(\Pi_2 \otimes |\delta_{P_H}|^s)$ with $s \in \mathbf{C}$. From this we can form a family of Einsenstein series $\mathscr{E}_s(\Pi_2, x)$ on the group $H(\mathbf{A})$. Then we form the integral

$$\int_{G_1(k)\backslash G_1(\mathbf{A})} f_{\Pi_1}(g_1)\mathscr{E}_s(\Pi_2, g_1)\, dg_1.$$

The problem is to determine when such an integral represents a good *L* function, i.e., one that is an Euler product and is related to Π_1 and Π_2 in a simple way. In any event, the above integral as a function in s has a meromorphic continuation and a functional equation with a precise location of poles. This follows from knowing the same information about the Eisenstein series $\mathscr{E}_s(\Pi_2, \ )$; this in turn follows from the Langlands–Shahidi method mentioned above. In any case the analytic *L* function that arises from the integral should be more *complicated* than the *L* function that arises in normalizing the Eisenstein series $\mathscr{E}_s(\Pi_2, \ )$.

In any event the Langlands–Shahidi method determines a prescribed list of *L* functions; this list does not include all the possible cases of *L* functions in [L]. The Rankin–Selberg method has so far yielded only some of the *L* functions that come about by the first method. Thus, one of the major question is whether the Rankin–Selberg method will give new examples of *L* functions not possible by Langlands–Shahidi method. We show that, in fact, this is the case. We construct here, via the Rankin–Selberg method, the *L* function associated to the tensor product of the standard 7-dimensional representation of G_2 with the standard two-dimensional representation of GL_2. In fact, as a bonus of our method, we also represent the *L* function of the standard 7-dimensional representation of G_2!

1. DATA OF PROBLEM

Let k be a field of characteristic 0. Let $\Lambda^*(k^m)$ be the exterior algebra over the *m*-dimensional column space k^m. We consider, specifically, the case $m = 7$ and the subspace $\Lambda^3(k^7)$. The group $GL_7(k)$ acts on k^7 (thru left multiplication of matrices). We consider the action of GL_7 on Λ^* and, in particular, on $\Lambda^3(k^7)$.

We say a form $\psi \in \Lambda^3$ is *nondegenerate* if ψ has the following property: For all $y \in k^7 - 0$, $\psi_y = \psi(y, *, *)$, (viewed as an element in $\Lambda^2(k^7/k \cdot y))$, is a nondegenerate 2 form. The set of all *nondegenerate* forms in Λ^3 is a stable under GL_7, and the stabilizer of any such form is a group isomorphic to $G \times Z_3^*$ with G, a group isomorphic to a form of G_2, where the group $Z_3^* = \mathbf{Z}/3\mathbf{Z}$ or 1 depending on whether k contains cube roots of identity. Such a ψ as above determines a symmetric form on k^7 via the following recipe: $\psi \wedge \psi_y \wedge \psi_z$ is a multiple of a fixed element ω_0 in $\Lambda^7(k^7)$, i.e. $\psi \wedge \psi_y \wedge \psi_z = \varphi_\psi(y, z)\omega_0$. Then φ_ψ determines a nondegenerate symmetric form on k^7 and $G \subseteq \text{Stab}_{GL_7}(\psi)$ leaves φ_ψ invariant. Thus $G \subseteq SO(\varphi_\psi) =$ the special orthogonal group of φ_ψ. (See [H] for these facts.)

If k is either a number field or a local field then there are at most two possibilities for φ_ψ. Either φ_ψ is anisotropic or φ_ψ represents a quadratic form of Witt index 3.

We let $X_l(\varphi_\psi) =$ the set of all isotropic subspaces of dimension equal to l relative to the form φ_ψ. The group G acts almost transitively on $X_l(\varphi_\psi)$ in the following sense.

Lemma 1.1. *If $l = 1$, then G acts transitively in $X_1(\varphi_\psi)$. If $l = 2$ or $l = 3$, G has a finite number of orbits in $X_l(\varphi_\psi)$.*

Remark. We note that if k is algebraically closed, the group G has a finite number of orbits acting on the space $B\backslash SO(\varphi_\psi)$, where B is a Borel subgroup of $SO(\varphi_\psi)$. We also note the case $l = 1$ is well-known, and a proof is given in [RS]. In the case $l = 2$ there are two orbits.

In the case $l = 2$, the *small* orbit has as representative stabilizer in G, a parabolic subgroup Q of G. The *large* orbit has as representative stabilizer, a subgroup H, which is given as a semidirect product $GL_2(k) \ltimes Z$ where Z is a 3-dimensional unipotent subgroup.

We know that $X_2(\varphi) \cong \widetilde{P}\backslash SO(\varphi_\psi)$ where $\widetilde{P}$ is a parabolic subgroup of $SO(\varphi_\psi)$ having the form $GL_2(k) \times SO(3) \ltimes U$ with U the 7-dimensional unipotent radical. Then we have that $H = G \cap \widetilde{P}$. The structure of H above is compatible with $\widetilde{P}$ in the following sense. First $Z \subseteq SO(3) \ltimes U$ and the $GL_2(k)$ subgroup of H is isomorphic to the embedding of $GL_2(k)$ into $GL_2(k) \times SO(3)$

given by the map

$$g \rightsquigarrow (g,\, i(\det g)),$$

and i is a canonical mapping of $k^\times$ into a split torus of $SO(3)$.

2. GLOBAL RANKIN–SELBERG INTEGRALS

We let k be a number field and $k_v = $ the local component associated to each prime v in k. Given an algebraic object X (i.e. group or affine variety) defined over k, we let $X(\mathbf{A})$ be the corresponding adelized space.

We consider the embedding of G into $SO(\varphi_\psi)$ given in §1. We recall that $\widetilde{P}$ is parabolic of $SO(\varphi_\psi)$ with Levi factor $GL_2 \times SO(3)$. We form the corresponding adelized groups $G(\mathbf{A})$, $SO(\varphi_\psi)(\mathbf{A})$, $\widetilde{P}(\mathbf{A})$, etc.

We consider the family of characters given on $\widetilde{P}(\mathbf{A})$ via the map $g = (x,\, y,\, u) \overset{\shortparallel}{\rightsquigarrow} |\det x|_{\mathbf{A}}^{s+2}$ with $x \in GL_2(\mathbf{A})$, $y \in SO(3)(\mathbf{A})$, $u \in \widetilde{U}(\mathbf{A})$. Moreover, we let τ be a cuspidal irreducible representation of $GL_2(\mathbf{A})$ occuring in $L^2(GL_2(k)\widetilde{Z}(\mathbf{A})\backslash GL_2(\mathbf{A}))$ with $\widetilde{Z} = $ the center of $GL_2(k)$. This means that τ has trivial central character. Then we form the family of induced automorphic representations

$$I(\tau,\, s) = \mathrm{Ind}_{\widetilde{P}(\mathbf{A})}^{SO(Q)(\mathbf{A})}(\tau \otimes |\,|^{s+2} \otimes 1_{SO(3)} \otimes 1_{\widetilde{U}}).$$

With such a family we can construct Eisenstein series. Let $f(\tau,\, s,\, x)$ be an analytic family in $I(\tau,\, s)$. Then for s so that $\mathrm{Re}(s)$ is sufficiently large we form the Eisenstein series

$$\mathscr{E}(f(\tau,\, s,\, x)) = \sum_{\widetilde{P}(k)\backslash SO(Q)(k)\ni\gamma} f(\tau,\, s,\, \gamma x).$$

Then $\mathscr{E}(\ )$ has a meromorphic continuation in s and admits a functional equation in s. For this we consider the $SO(Q)(\mathbf{A})$ intertwining operator $M_w \colon I(\tau,\, s) \to I(\tau,\, -s)$ defined by

$$M_w(f(\tau,\, s,\, x)) = \int_{\widetilde{U}(\mathbf{A})} f(\tau,\, s,\, wux)\,du,$$

(with w, the appropriate Weyl group element which sends all the positive roots to negative roots). Then the functional equation of $\mathscr{E}(f(\tau,\, s,\,))$ is

$$\mathscr{E}(f(\tau,\, s,\, x)) = \mathscr{E}(M_w(f(\tau,\, s,\, x))).$$

Moreover, if Ω is any finite set of primes, we denote by the restricted L^{Ω} function as the Euler product

$$\prod_{v \notin \Omega} L_v(\cdots)$$

of the appropriate local $L_v(\cdots)$ factors. Since the representation $I(\tau, s) = \bigotimes_v I_v(\tau_v, s)$ with $I_v(\tau_v, s)$, the corresponding local representation, we say $f(\tau, s, \ )$ is factorizable if $f(\tau, s, \ ) = \prod_v f_v(\tau_v, s, \ )$. Let Ω' be the set of primes where $f_v(\tau_v, s, \ )$ is not the unique spherical vector in $I_v(\tau_v, s)$. Then $f(\tau, s, \ ) = f^{\Omega'}(\tau, s, \ )f_{\Omega'}(\tau, s, \ )$ when $f^{\Omega'} = \prod_{v \notin \Omega} f_v(\tau_v, s, \ )$ and $f_{\Omega'} = \prod_{v \in \Omega} f_v(\tau_v, s, \ )$. In particular we know that

$$M_w(f(\tau, s, \ )) = \left[\frac{L^{\Omega'}(\tau, s-1/2)}{L^{\Omega'}(\tau, s+3/2)}\right]\left[\frac{L^{\Omega'}(\tau, \mathrm{Sym}^2, 2s)}{L^{\Omega'}(\tau, \mathrm{Sym}^2, 2s+1)}\right]$$
$$\times f^{\Omega'}(\tau, -s, \ )\prod_{v \in \Omega}(M_w^v)(f_v(\tau_v, s, \ ))$$

where M_w^v is the corresponding local intertwining operator associated to M_w. Here $L^{\Omega}(\tau, s)$ and $L^{\Omega}(\tau, \mathrm{Sym}^2, s)$ are the restricted L functions determined by the standard 2-and 3-dimensional representations of $GL_2(\mathbf{C})$ associated to the cuspidal representation τ of $GL_2(\mathbf{A})$. Then we expect from the theory of Eisenstein series that

$$s \rightsquigarrow L^{\Omega'}(\tau, s+3/2)L^{\Omega'}(\tau, \mathrm{Sym}^2, 2s+1)\mathscr{E}(f(\tau, s, x))$$

admits a finite number of poles.

Let Π be a cuspidal irreducible representation of $G(\mathbf{A})$. Let $F \in \Pi$ be a smooth function. We form the Rankin–Selberg integral:

$$\mathrm{II}(F, f(\tau, s, \ )) \equiv L^S(\tau, s+3/2)L^S(\tau, \mathrm{Sym}^2, 2s+1)$$
$$\times \int_{G(k)\backslash G(\mathbf{A})} F(x)\mathscr{E}(f(\tau, s, x))\, dx.$$

In this case the set S depends on F and $f(\tau, s, \ )$ to be specified below.

We let U_0 be the unipotent radical of a Borel subgroup of G. We consider an additive character ψ_{U_0} on the $U_0(k)\backslash U_0(\mathbf{A})$ which has the property that it is nontrivial on each root subgroup of $U_0(\mathbf{A})$ which corresponds to a simple root. Then we say Π is a

generic representation of $GL(\mathbf{A})$ if the integral

$$\int_{U_0(k)\backslash U_0(\mathbf{A})} F(u)\psi_{U_0}(u)\,du$$

is nonzero for some $F \in \Pi$.

Given $F \in \Pi$ we say F is *factorizable* in the following sense. There exists a $G(\mathbf{A})$ intertwining map $r_\Pi\colon \bigotimes_v \Pi_v \to \Pi$ and let $r_\Pi(\bigotimes_v F_v) = F$ where F_v for almost all v is the unique G_v spherical vector in Π_v.

We let $L_v\,(\Pi_v \boxtimes \tau_v, s)$ be the local Euler factor defined in the following manner. The representation Π_v (τ_v resp.) corresponds to a semisimple element A_v in $^L G^0 = G_2(\mathbf{C})$ (B_v in $^L GL_2^0 = GL_2(\mathbf{C})$ resp.) determined by the Satake classification of spherical representations of G. (GL_2 resp.). Then $L_v(\Pi_v \boxtimes \tau_v, s)$ is defined as

$$[\det(I_{14} - A_v \otimes B_v q_v^{-s})]^{-1}.$$

Theorem 2.1. *Let* $f(\tau, s, \ )$ *and* F *be factorizable functions as defined above.*

Let S *be the set of places* v *where* v *is Archimedean and where the local components* $f_v(\tau_v, s, \ )$ *and* F_v *are not spherical vectors.*

Let Π *be a "generic" representation. Then*

$$\mathrm{II}\left(F, f(\tau, s, \ )\right) = \mathrm{II}_S(F, f(\tau, s))L^S\left(\Pi \boxtimes \tau, \frac{1}{2} + s\right),$$

where II_S *is a local Euler product (specified below) over the primes* $v \in S$ *and* $L^S(\Pi \boxtimes \tau, 1/2 + s)$ *is the restricted* L *function*

$$L^S(\Pi \boxtimes \tau, 1/2 + s) = \prod_{v \notin S} L_v(\Pi_v \boxtimes \tau_v, 1/2 + s).$$

Moreover the function $L^S(\Pi \boxtimes \tau, s)$ *admits a meromorphic extension to* $\mathbf{C}$.

Remark. Since the Eisenstein series $\mathscr{E}(f(\tau, s, \ ))$ is normalized by $L^S(\tau, s + 3/2)L^S(\tau, \mathrm{Sym}^2, 2s + 1)$ then it should follow that the function $s \rightsquigarrow L^S(\Pi \boxtimes \tau, s)$ admits a finite number of poles!

By the well-known principle of uniqueness of Whittaker models the representation Π_v can be embedded into $\mathrm{Ind}_{(U_0)_v}^{G_v}(\psi_{(U_0)v})$. Similarly τ_v can be embedded into $\mathrm{Ind}_{N(k_v)}^{G_2(k_v)}(\psi_v)$ where ψ_v is a nontrivial additive character on $N = \{\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) | x \in k_v\}$. Using these embeddings we let $W_v(\tau_v, g, s)$ belong to $\mathrm{Ind}_{P_v}^{G_v}(\tau_v \otimes 1|_v^{s+2})$

and $W_{F_v} \in \Pi_v$ be the corresponding Whittaker functions. Then we form the integral

$$(*)_v(s) = \int_{V_v \backslash G_v} W_{F_v}(g) W_v(\tau_v, g, s) \, dg,$$

where V_v is the 4-dimensional subgroup of $(U_0)_v$ which is an extension of Z_v by the 1-dimensional root subgroup corresponding to a long simple root of G.

In fact we have that the data is chosen in such a way that

$$\mathrm{II}_S(F, f(\tau, s)) = \prod_{v \in S} (*)_v(s).$$

Then we prove that

Corollary to Theorem 2.1. *Let $s_0 \in \mathbf{C}$ be given. Then it is possible to choose the data F and $f(\tau, s, \ )$ (depending on s_0) so that $\mathrm{II}(F(f(\tau, s, \ )) = I_\infty(F, f(\tau, s, \ )) L^S(\Pi \boxtimes \tau, \frac{1}{2} + s)$ where I_∞ is a meromorphic function in s which is nonvanishing at s_0.*

We note that we can extend the application of the method of Theorem 2.1 to cover another L function for the group G. Rather than use a cusp form τ we use an Eisenstein series on the group $GL_2(\mathbf{A})$. Indeed we consider the parabolic subgroup $\widetilde{P}'$ of $SO(\varphi_\psi)$ contained in $\widetilde{P}$ of the form $\widetilde{B}_2 \cdot SO(3) \cdot \widetilde{U}$ where $\widetilde{B}_2$ is the Borel subgroup of GL_2. Then considering $T_2 \subset B_2$ as a maximal torus, we define a character $(\delta(s, \mu)$ on $T_2 = \{(t_1, t_2) \in k^x \times k^x\} \rightsquigarrow |t_1|^{s+\mu+5/2} |t_2|^{s-\mu+3/2}$. We form the family of induced automorphic representations

$$I(\mu, s) = \mathrm{Ind}_{\widetilde{P}'(\mathbf{A})}^{SO(Q)(\mathbf{A})} (\delta_{(s, \mu)} \otimes 1_{SO(3)} \otimes 1_{\widetilde{U}}).$$

With such a family we construct Eisenstein series. Let $f(s, \mu, x)$ be an analytic family in $I(\mu, s)$, and we form for appropriate (s, μ) the Eisenstein series

$$\mathscr{E}(f(s, \mu, x)) = \sum_{\widetilde{P}'(k) \backslash SO(Q)(k)} f(s, \mu, \gamma x).$$

Again the normalizing factor for $\mathscr{E}(f(s, \mu, x))$ is

$$L^S(1, s + \mu + 3/2) L^S(1, s - \mu + 3/2) L^S(1, s + \mu + 1/2)$$
$$\times L^S(1, s - \mu + 1/2) L^S(1, 2s + 2\mu) L^S(1, 2s - 2\mu)$$
$$\times L^S(1, 2s + 1) L^S(1, 2\mu + 1)$$
$$\equiv \xi^S(s, \mu).$$

Then we form again the Rankin–Selberg integral

$$\mathrm{II}(F, f(s, \mu,)) = \xi^S(s, \mu) \int_{G(k)\backslash G(\mathbf{A})} F(x)\mathscr{E}(f(s, \mu, x))\, dx.$$

Then we prove the analogue of Theorem 2.1.

Theorem 2.2. *Let F and $f(s, \mu,)$ be factorizable functions as given in Theorem* 2.1. *Let S be also as given in Theorem* 2.1. *Then*

$$\mathrm{II}(F, f(s, \mu,)) \equiv \mathrm{II}_S(F, f(s, \mu,))$$

$$L^S(\Pi, s + \mu + 1/2)L^S(\Pi, s - \mu + 1/2)$$

where II_S is a local Euler product over the primes in S and $L^S(\Pi, \lambda)$ is the restricted L function

$$L^S(\Pi, \lambda) = \prod_{v \notin S} L_v(\Pi_v, \lambda).$$

Here L_v is the local Euler factor defined in the following manner (see comments preceding Theorem 2.1)

$$L_v(\Pi_v, \lambda) = [\det(I_7 - A_v q_v^{-\lambda})]^{-1}.$$

Moreover the function $L^S(\Pi, \lambda)$ admits a meromorphic extension to $\mathbf{C}$.

Remark. We note that we can specialize the above Rankin–Selberg integral along a certain hyperplane $s - \mu + 1/2 = K_0$ (along which the Eisenstein series $\mathscr{E}(f(s, \mu,))$ has no singularities) with K_0 a real large integer. Then we have a Rankin–Selberg representation of $L^S(\Pi, \lambda)$. Note that $L^S(\Pi, s - \mu + 1/2) = L^S(\Pi, K_0)$ is given by a convergent Euler expansion! We also expect as before that $L^S(\Pi, \lambda)$ has a finite number of poles in λ.

3. UNRAMIFIED CASE AND BRANCHING FORMULA

If the data is $(*)_v$ is unramified (i.e. the local components are spherical) then we calculate

Lemma 3.1.

$$(*)_v(s) = \frac{L_v(\Pi_v \boxtimes \tau_v, \frac{1}{2} + s)}{L_v(\tau_v, Sym^2, 2sH)L_v(\tau_v, s + \frac{3}{2})}.$$

The basis of such a calculation is the use of the Casselman–Shalika formula for Whittaker functions in terms of characters of irreducible finite-dimensional representations.

The critical step in this calculation is the use of a certain Branching Formula from $SO(7, \mathbf{C})$ with highest weight $(\lambda_1, \lambda_2, 0)$ (with $\lambda_i \in \mathbf{Z}$ and $\lambda_1 \geq \lambda_2 \geq 0$) and consider the restriction to $G_2(\mathbf{C})$.

We parametrize a highest weight representation of $G_2(\mathbf{C})$ by a pair (m, n) (with $m, n \in \mathbf{Z}$ and $m \geq n \geq 0$), which corresponds to the weight $n\omega_1 + (m - n)\omega_2$ with ω_1, ω_2 fundamental weights of $G_2(\mathbf{C})$ dual to the two simple roots α_1 and α_2. Then the particular Branching Formula we prove is the following:

$$(\lambda_1, \lambda_2, 0) = \sum_{\alpha=0}^{\alpha=\min(\lambda_1-\lambda_2,\lambda_2)} \sum_{\beta=\alpha}^{\beta=\lambda_2} (\lambda_1 - \alpha, \beta).$$

The formula is stated in [J]. Our proof is based on using the ideas of [BGG] and [KV].

REFERENCES

[BGG]   J. Bernstein, I. Gelfand, and S. Gelfand, *Models of representations of Lie groups*, Selecta Math. Soviet **1** (2) (1981), 121–142.

[G-PS-R] S. Gelbart, I. Piatetski-Shapiro, and S. Rallis, *Explicit constructions of automorphic L functions*, Lecture Notes in Math., vol. 1254, Springer-Verlag, New York, 1989.

[GS]   S. Gelbart and F. Shahidi, *Analytic properties of automorphic L functions*, Perspectives in Mathematics, Academic Press, Boston, Massachusetts, 1988.

[H]   C. Herz, *Alternating 3 forms and exceptional Lie groups of type G_2*, Canad. Math. J. **35** (1983), 776–806.

[J-L]   H. Jacquet and R. Langlands, *Automorphic forms on GL(2)*, Lecture Notes in Math., vol. 104, Springer-Verlag, New York, 1970.

[J]   B. R. Judd, *Atomic spectroscopic methods*, Harcourt Brace, New York 1977.

[J-PS-Sh] H. Jacquet, I. Piatetski-Shapiro, and J. Shalika, *Rankin–Selberg convolutions*, Amer. J. Math., **105** (1983), 367–464.

[J-S-1,2] H. Jacquet and J. Shalika, *On Euler products and classification of automorphic representations,* I *and* II, Amer. J. Math. **103** (3) (1981), 499–558 and 777–815.

[KV]   B. Kimelfeld and E. Vinberg, *Homogeneous domains on flag manifolds and spherical subgroups of semisimple Lie groups*, Functional Anal. Appl. **12** (3) (1978), 168–174.

[L]   R. Langlands, *Euler products*, Whitmore Lectures, Yale Univ. Press, 1967.

[PS-R] I. Piatetski-Shapiro and S. Rallis, *Rankin triple L-functions*, Comp. Math. **64** (1987), 31–115.

[RS]   S. Rallis and G. Schiffman, *Theta correspondence associated to G_2*, Amer. J. Math. (to appear).

[W-1]   A. Weil, *Basic number theory*, die Grundlehren der Mathematischen Wissenschaflen, vol. 144, Springer-Verlag, Berlin and New York, 1967.

[W-2]   A. Weil, *Uber die Bestimmung Dirichletscher Reihen durch funktionalgleichungen*, Math. Ann. **168** (1967), 149–156.

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