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Theory of relations, by R. Fraïssé. Studies in Logic and the Foundations of Mathematics, vol. 118, North-Holland, Amsterdam, 1986, xii+397 pp., $55.25. ISBN 0-444-87865-3

What in the world is the theory of relations? I am not sure, but I know that I like it. What are the simplest relations on a set? Most mathematicians would include partial orders and linear orders. The theory of relations is concerned with these (as well as the more general n-ary relations) and basic notions such as embeddability and isomorphism. The subject originated with Hausdorff (*Grundzüze der Megenlehre*, 1914) and Sierpiński (*Leçons sur les nombres transfinis*, 1928).

One of my favorite topics is the theory of betterquasiorders. A quasiorder is a binary relation on a set which is reflexive and transitive, but not necessary antisymmetric. A quasiorder $\leq$ becomes a partial order if we mod out by the equivalence relation $\approx$ defined by $x \approx y$ iff $x \leq y$ and $y \leq x$. A good example of a quasiorder is to take the family of linear orders and put $L_1 \leq L_2$ iff there exists an embedding of L_1 into L_2, i.e., one-to-one mapping $f: L_1 \mapsto L_2$ such that for all x, $y \in L_1$ we have $x \leq_1 y$ iff $f(x) \leq_2 f(y)$.

One of the more interesting properties a quasiorder can have is that of being a wellquasiorder. A quasiorder $(X, \leq)$ is a wellquasiorder iff for any sequence $\langle x_n : n \in N\rangle$ of elements of X there exists some $n < m$ such that $x_n \leq x_m$. Clearly a wellquasiorder cannot have an infinite strictly descending sequence nor an infinite sequence of incomparable elements. Surpisingly these two properties characterize wellquasiorders. This is a nice exercise using Ramsey's theorem.

On of the first theorems about wellquasiorderings is due to Kruskal (1960). This theorem says that the finite trees are wellquasiordered under embeddability. A tree is partial order $(T, \leq)$ such that every initial segment, i.e., set of the form $\{y \in T: y \leq x\}$, is linearly ordered by $\leq$.

In 1948 Fräissé conjectured that the family of countable linear orders is wellquasiordered under the quasiorder of embeddability. Laver (1971) was able to prove this conjecture. One interesting thing about the proof is that it requires a stronger notion called betterquasiorder. This notion was invented by Nash–Williams (1968) in order to prove the following theorem. Given a quasiorder P we can define a quasiorder $\leq$ on P^{ω} the countable infinite product of P by defining $\langle a_n : n \in \omega\rangle \leq \langle b_n : n \in \omega\rangle$ iff there exists a subsequence $\langle k_n : n \in \omega\rangle$ such that for all n we have $a_n \leq b_{k_n}$. Rado (1954) showed that wellquasiorderedness need not be preserved by such infinite products. Nash–Williams showed that the stronger property of betterquasiordered is preserved by such infinite products. (In fact, one of the equivalents of betterquasiorder is that the ordinal product is wellquasiordered for every ordinal.)

Let's get back to Fräissé's conjecture. Cantor proved that every countable linear order can be embedded into the rationals. It suffices to restrict our attention to the countable scattered linear orders, i.e., those which do not contain an isomorphic copy of the rationals, since any two nonscattered countable orders can be embedded into each other. In 1914 Hausdorff analyzed the countable scattered linear orders by giving a construction technique. He showed that the family of countable scattered linear orders is the smallest class closed under isomorphism which contains the finite linear orders, and closed under finite sums, ω-sums, and converses (i.e. the reverse order). The sum of two linear orders L_1 and L_2 is defined by taking the union of disjoint copies of L_1 and L_2 and letting everything in L_1 be less than everything in L_2. The ω-sum of a sequence of linear orders $\langle L_n : n \in \omega\rangle$ is defined by taking the union of disjoint copies of the L_n and for any $n < m$ everything in L_n be less than letting everything in L_m. This construction principal naturally leads to a hierarchy of scattered linear orders, indexed by the set of countable ordinals. Namely let $\mathscr{H}_0$ be the one element linear orders, and for any ordinal $\alpha > 0$ let $\mathscr{H}_\alpha$ be all linear orders which are finite sums, ω-sums, or converses of linear orders from $\bigcup_{\beta<\alpha} H_\beta$. One of the needs for betterquasiorder theory is to handle the problem of ω-sums.

Nash–Williams betterquasiorder theory is also closely connected to the Galvin–Prikry theorem (1973). Work of Steve Simpson (see Chapter 9 of *Recursive aspects of descriptive set theory*, by R. Mansfield and Galen Weitkamp, Oxford University Press, 1985) makes this theory even closer connected.

Let $\omega = \{0, 1, 2, \ldots\}$ and for any set X let $[X]^{\omega}$ be the family of countably infinite subsets of X. The basic open sets of the topology on $[\omega]^\omega$ are of the form $\{x \in [\omega]^\omega : X \cap n = F\}$, where $n \in \omega$ and $F \subset \omega$ is finite.

The Galvin–Prikry theorem says that for any Borel subset B of $[\omega]^\omega$ and any $X \in [\omega]^\omega$ there exists $H \in [X]^\omega$ such that either $[H]^\omega \subset B$ or $[H]^\omega \cap B = \varnothing$. This is kind of an infinite analogue of Ramsey's theorem. Simpson has shown that a quasiorder Q is betterquasiordered iff for every Borel map $f: [\omega]^\omega \mapsto Q$ (where Q has the discrete topology) there exists an $X \in [\omega]^\omega$ such that $f(X^*) \leq f(X)$ where X^* is X minus its minimum element.

I would like to mention some other interesting theorems in the theory of relations. Szplirajn (1930) showed that every partial-order P can be extended to a linear-order L. It is also well known that every well-founded partial order (i.e. one in which there does not exist an infinite descending sequence) can be extended to a wellordering. Bonnet and Pouzet (1969) showed that for any partially ordered set P if the rationals cannot be embedded into P, then there exists a linear-order L extending P such that the rationals cannot be embedded into L.

There are many other interesting theorems in the theory of relations not only about partially ordered sets but also about more general n-ary relations. Fräissé's book contains too many of them to mention here, but I hope I have given the reader a taste of the subject.

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ARNOLD W. MILLER
UNIVERSITY OF WISCONSIN

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Boundary value problems of finite elasticity, by T. Valent. Springer-Verlag, Heidelberg, 1988, xii+192 pp., $64.00. ISBN 0-387-96550-5

§1. Introduction

During the past decades the *mathematical* theory of nonlinear three-dimensional elasticity has undergone a considerable renewed interest, reflected for instance by the books of Marsden and Hughes [17], Ciarlet [8], and the book reviewed here.

The existence results available at the present time fall in two categories:

In one approach (described in §§2 and 5) the problem is posed as a *system of three quasilinear partial differential equations of the second order*, together with specific boundary conditions (cf. (13)), and one tries to obtain "*local*" *existence results based on the implicit function theorem*; this approach, which was initiated by Stoppelli [18], is the central theme of the book under review.

In another approach (described in §§3 and 4), the problem is posed as a *minimization problem for the associated energy* (cf. (20)), and one tries to adapt the paraphernalia of the calculus of variations (infimizing sequences, weak convergence, weak lower semi-continuity, etc.) to this problem, which is "highly nonconvex"; this approach is the basis of a famous existence result of Ball [3].

All these results apply to "static" equilibria, i.e. to problems that are *time-independent*. While substantial progress has thus been made in the study of statics, the mathematical analysis of time-dependent three-dimensional elasticity still meets with inextricable difficulties. The proofs of the available existence results "for large