BOOK REVIEWS

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Schander bases: Behavior and stability, by P. K. Kamthan and M. Gupta. Pitman Monographs in Pure and Applied Mathematics, Longman Scientific and Technical, New York, 514 pp., $79.95. ISBN 0-470-21029-X

1. Basis

A sequence (x_n) in a Banach space X is a *Schauder basis* provided that for every $x \in X$ there is a unique sequence of scalars (a_i) such that

$$x = \sum_{n=1}^{\infty} a_n x_n$$

convergence in the norm topology.

This notion, obviously generalizing the classical notion of basis for a finite dimensional vector space, is due to Schauder [S]. Schauder bases (and certain generalizations of the notion) have played an important role in developing general Banach space theory.

2. Banach

Banach thought enough of the topic (Schauder bases) to devote Chapter VII of his classic monograph *Théorie des opérations linéaires* to the subject. (Albeit a short—six pages—chapter. But all the chapters in this classic work are short!)

Obviously, the existence of a basis imposes structure not enjoyed by all Banach spaces. It is truly trivial that if X has a basis then X must be separable. Perhaps the most important question ever raised about bases—Does every separable Banach space have a Schauder basis?—was posed by Banach himself [B, p. 111]. This problem, the so-called *Basis problem*, eluded researchers for years but was finally settled, negatively, by Per Enflo [E] in 1973. (Actually, Enflo solved the larger "approximation Problem.") Thus, there are natural questions about bases.

What more can be said about a basis? If the Banach space X has a Schauder basis (x_n) and $x = \sum_{n=1}^{\infty} a_n x_n$ is the unique basis expansion of $x \in X$, define $f_i(x) = a_i$. Then f_i is clearly linear and $f_i(x_i) = \delta_{ij}$ (0 if $i \neq j$; 1 if $i = j$).

In his original definition of basis, Schauder *assumed* continuity of the f_i's. This innocuous assumption, in some sense, set the stage for all that followed. By a clever application of the Banach-Steinhaus theorem, Banach proved (*in twelve lines*) [B, p. 111] that the f_i's are automatically continuous. These twelve lines contain ideas which have generated thousands of pages of "basis theory." Here are Banach's ideas: If the Banach space X has a Schauder basis (x_n), let

$$\lambda = \left\{ (a_i) \middle| \sum_{n=1}^{\infty} a_n x_n \in X \right\}.$$

Using coordinate arithmetic, λ is a vector space of sequences. If λ is normed by

$$||(a_n)|| = \sup_m \left|\left| \sum_{n=1}^{m} a_n x_n \right|\right|,$$

then λ becomes a Banach space and the mapping

$$T\{(a_n)\} = \sum_{n=1}^{\infty} a_n x_n$$

is a linear homeomorphism from λ onto X (this part requires the completeness of λ). Since

$$||a_i x_i|| \leq 2||(a_i)||$$

it follows that

$$|f_i(x)| \leq 2||x_i||^{-1} ||T^{-1}|| \, ||x||.$$

Idea 1. *If X has a basis, X can be viewed as a sequence space.*

It was also observed by Banach (using different terminology of course) [B, p. 107] that if (x_n) is a Schauder basis for X, and (f_n) is defined as above, then (f_n) is a basis for the conjugate space X^* equipped with the w^*-topology. That is, for every $f \in X^*$ and $x \in X$,

$$f(x) = \sum_{n=1}^{\infty} f(x_n) f_n(x).$$

Idea 2. *Schauder bases exist (and arise naturally) in certain nonnormable, even nonmetrizable spaces.*

Two remarks of Banach [B, p. 238] (appropriately in the "Remarques au Chapitre VII") reinforce these ideas.

"La notion de base peut être évidentment introduite d'une façon plus génerale déjà pour les espaces de type (F). Dans l'espace (s) (Reviewer: the space of all real sequences with the product topology) la base est donnée p. ex par la suite d'élements (x_i) ou

$$x_i = (\zeta_n^i) \quad \text{et} \quad \zeta_n^i = \begin{bmatrix} 1 \text{ pour } i = n \\ 0 \text{ pour } i \neq n \end{bmatrix}.$$

(Reviewer's emphasis). *Fréchet spaces are a good place to look, among the classical nonnormable spaces, for Schauder bases.*

Back, to Banach:

"La espace $S[0, 1]$ (Reviewer: the space of measurable real-valued functions on $[0, 1]$ with the topology of convergence in measure), ne contient aucune base: c'est une consequence du fait gu'il n'y existe aucune fonctionnelle linéaire ne's annulant pas identiquement."

(Reviewer's emphasis). *One wants (needs) continuous linear functionals for basis expansions. Thus, stay away from nonlocally convex spaces.*

3. Book

The authors of the book under review attack these Banachian ideas with a vengence. Page 13 (book) lists fourteen sequence spaces used throughout the text and many, many more appear. Remember: If you have a basis you have a sequence space. Part II (242 pp.) deals with types of bases and the associated sequence spaces. Here is the idea: Given a basis (x_n) with coefficient functionals (f_n) in a locally convex space, one places topological properties on (x_n) or (f_n) (e.g., bounded, bounded away from zero, $(\sum_{m=1}^n x_m)$ bounded, equicontinuity of the partial sum operators $S_n(x) = \sum_{m=1}^n f_m(x)x_m$, etc.) or on the mode of convergence of $\sum_{n=1}^\infty a_n x_n$ (e.g., unconditional, absolute) and then tries to see what conditions these properties impose on the associated sequence space λ defined á la Banach above.

Part III (175 pp.) deals with similar bases and stability. Bases (x_n) and (y_n) are *similar* if the basis to basis map

$$Tx_n = y_n$$

is an isomorphism. Clearly, similar bases give rise to the same sequence space. *Stability* refers to the following: If (x_n) is a basis and (y_n) is "near" (x_n), does it follow that (y_n) is also a basis? Various nearness criterion are discussed (see below).

Part IV (117 pp.) concerns bases in nuclear spaces. The definition of nuclear space, involving "nuclear bonding maps" is rather technical and need not concern us here. Suffice it to say that the nuclear Fréchet spaces include most of the important nonnormable Fréchet spaces. In many ways, nuclear spaces behave like finite dimensional spaces and their "basis structure" is therefore much different from that of the basis structure of other locally convex spaces. These differences are discussed in Part IV.

Part I is a review of locally convex theory and is thus not important to our discussion.

4. Ballyhoo

From the preface:

> The aim of the authors is two fold: "to collect systematically more or less all the significant work on Schauder basis theory in locally convex spaces ... and communicate this to graduate and advanced undergraduates with the sole intention of generating interest in the field and secondly, to provide for experts a single source of reference containing information on the heuristic development of the subject matter in this book. [KG3]

From the blurb [KG3]:

> The lucid and systematic treatment of the subject matter will help beginners overcome their initial difficulties. To the more advanced workers, the book offers an in-depth treatment of some of the more difficult aspects of the theory, for example, bases and nuclearity of spaces. To the experts, this volume offers the advantages of having the theory brought together into a single work of reference.

5. Bias

The authors claim that they wrote the book because [KG3]: "A need has been felt for several years for a book that would cover, in depth, the theory of Schauder bases in locally convex spaces."

The reviewer's not quite favorable opinion of the book is based in part on the belief that they waited too long: The book under review would have been a pretty good book—if it had been written approximately twenty years ago! In fairness, this is the third volume issued from a project begun in 1972. However, the bibliography for the present volume contains references as late as 1986 and several "in preparation." Consider the bibliography: There are 242 items listed. Of these, 137 are before 1970 (56.6%). Not too bad. However, the authors refer to themselves 50 times and only two of their papers (listed) appeared before 1970 (and these two have little to do with Schauder bases). Of the 28 papers listed which appeared *after* 1980, 24 are to the authors and their school.

While this certainly indicates that the authors felt a need for this work, the evidence of the bibliography indicates (to the reviewer anyway) that this need was not felt by the general mathematics community nor specialists in Functional Analysis.

Another reason for this twenty years ago feeling is the last part of the book dealing with bases in nuclear spaces. Much of the material here is concerned with theorems of Dragilev and Wojtynski. These are important contributions to the theory of nuclear spaces. We skip the content and suffice it to say that the work of Dragilev appeared in 1960, 1965, and 1969, [D1,2,3]. That of Wojtynski appeared in 1970 [W]. Of course, some extensions have been made since 1970 but, basically, the theory has remained unchanged.

6. Blues

It is difficult to criticize a subject that was dear to the reviewer's heart (twenty years ago, of course). It is also difficult to criticize a book in which the reviewer's work (of twenty years ago) is treated in a complimentary fashion. Part of my problem with the present volume is given above. However, I have (at least) two other biases. One is viewpoint. That of the authors is contained in their remark [KG3, p. 326]:

"It is trivially seen that these results envelop the corresponding ones on Banach and F-spaces."

They are here referring to Paley-Weiner type theorems (see below), but it is essentially their viewpoint for the entire work.

The reviewer's viewpoint (admittedly changed over the years) is exactly the opposite: It is usually easy to take the known results concerning bases in Banach and Fréchet spaces and re-work them in a suitable (locally convex) setting. Of course there are exceptions. However, let us look at the Paley-Weiner theorem which motivates a large part of the section on stability.

The Paley-Weiner Theorem for Banach spaces: Let (x_n) *and* (y_n) *be sequences in a Banach space* X *and suppose there is a* μ, $0 < \mu < 1$ *such that*

$$\left\|\sum_{i=1}^{n} a_i(x_i - y_i)\right\| \leq \mu\left\|\sum_{i=1}^{n} a_i x_i\right\|$$

holds for all scalars (a_i) *and any* n. *Then most any reasonable property of* (x_n) *is inherited by* (y_n).

I am, of course paraphrasing but one should get the gist of the result. In particular if (x_n) is a basis so is (y_n).

The Paley-Weiner theorem for locally convex spaces [KG3, p. 327]: *Let* X *be a complete locally convex space and* (x_n), (y_n) *sequences in* X. *Suppose that for every continuous seminorm* p *on* X *there is a* μ_p, $0 < \mu_p < 1$ *such that*

$$p\left(\sum_{i=1}^{n} a_i(x_i - y_i)\right) \mu_p p\left(\sum a_i x_i\right)$$

holds for all scalars (a_n) *and any* n. *Then any reasonable property of* (x_n) *is inherited by* (y_n).

My other bias is style. My first objection is the inordinate use of abbreviation. Some sixty-two abbreviations are listed at the beginning of the book and many more are scattered throughout the text. I opened the book, arbitrarily to p. 180. Here is Proposition 9.2.18 on that page: *Every s-* S.b $\{x_n, f_n\}$ *for an S-space* (X, T) *is an* a.s.S.b.

One *can* find out what this means. But, is all the abbreviation really necessary?

Another stylistic problem, as I see it, is that the book has no real direction. Like an encyclopedia, an assortment of facts is offered. There is no attempt to separate the important from the merely true, or the deep from the trivial.

Used as a reference tool (by experts) all three of the Kamthan-Gupta books [KG1,2,3] have some merit. Anything one wants to know about bases in locally convex spaces can probably be found in these volumes. However, I do not see any of these books as a useful way to get students into research. And, these nonexperts are my greatest concern. The intricacies of nuclear spaces or, more generally Kothe sequence spaces, are beyond the grasp of beginners, at least any I have encountered.

Thus, to reiterate, I see the book under review as a general reference book for experts and advanced students and like any general reference, the book has some value. The book contains a lot of material. Unfortunately, I felt that I had read most of it years before. In the words of the great philosopher Yogi Berra, reading this was déjà vu all over again.

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Continuous decoupling tranformations for linear boundary value problems, by P. M. van Loon. Centrum voor Wiskunde en Informatica, CWI Tract #52, Stichting Mathematisch Centrum, Amsterdam, 1988, vi + 198 pp. ISBN 90-6196-353-2

This book deserves more readers than its title is likely to attract. Only specialists who have already some familiarity with the