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RECENT CLASSIFICATION AND CHARACTERIZATION RESULTS IN GEOMETRIC TOPOLOGY

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ABSTRACT. We announce a complete topological classification of the function spaces $C_p(X)$ of Borel class not higher than 2, provided that X is a countable space. We also present a topological classification of the k-dimensional universal pseudoboundaries and pseudointeriors in $\mathbf{R}^n$, and we investigate under what conditions strong negligibility of σZ-sets characterizes Hilbert space manifolds.

1. INTRODUCTION

The work presented in this announcement traces its history back to Fréchet [18] and Banach [6] who proposed to classify metric linear spaces according to topological type. For complete spaces this program was carried out by Anderson [1], Kadec [21], and Toruńczyk [25]: A Fréchet space is characterized topologically by its linear dimension (i.e., minimal cardinality of sets with a dense span). The classification of incomplete linear spaces, however, is still in the beginning stage. In the case that the space is a so-called absorber (see §2) characterizations have been developed (Mogilski [24] and Bestvina and Mogilski [8]). We apply these results to the classification of certain function spaces. If X is a space, then $C_p(X)$ denotes the space of continuous, real valued functions on X endowed with the topology of pointwise convergence. This function space is metrizable only if X is a countable space (barring spaces without point separating real valued functions). Therefore we consider countable completely regular spaces X that are for obvious reasons also nondiscrete. We show that all $F_{\sigma\delta}$-spaces

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$C_p(X)$ are homeomorphic to σ^ω, the countable product of $l_f^2 = \{(x_i) \in l^2 : x_i = 0 \text{ for almost all } i\}$. According to Dijkstra et al. [14], $C_p(X)$ cannot be an absolute $G_{\delta\sigma}$-set and hence we obtain a complete classification of the spaces $C_p(X)$ of Borel class not higher than 2. Using similar techniques, we detect other sequence and function spaces homeomorphic to σ^ω.

It was observed by Geoghegan and Summerhill [20] that the techniques developed for infinite-dimensional manifolds can be applied to $\mathbf{R}^n$ as well. They constructed k-dimensional universal pseudoboundaries B_k^n and pseudointeriors s_k^n in $\mathbf{R}^n$ analogously to the pseudoboundary B and the pseudointerior s of the Hilbert cube. Pseudoboundaries are absorbers and pseudointeriors are their complements. The pseudointeriors s_k^n for $n \geq 2k+1$ are particularly interesting because of their resemblance to the separable Hilbert space l^2, which is homeomorphic to the pseudointerior s by a celebrated theorem of Anderson [1]. We show that B_k^n is homeomorphic to B_k^m if and only if s_k^n is homeomorphic to s_k^m if and only if $n = m$ or $n, m \geq 2k + 1$.

In the final section we consider Hilbert space manifolds and strongly negligible sets. A strongly negligible subset of a space is a set whose complement is topologically equivalent to the whole space via homeomorphisms that can be chosen arbitrarily close to the identity. This concept is closely related to absorbers (see Chapter 1 in [9]). Anderson shows in [2] that in Hilbert space manifolds the strongly negligible sets coincide with the σZ-sets. We discuss the result that this property characterizes Hilbert space manifolds if every compactum is a strong Z-set but not if every compactum is merely a Z-set.

For background information on infinite-dimensional topology see [7] or [23].

2. Absorbers

The central idea in this note is the concept of a (generalized) absorber. Let X be a space and let $\mathscr{M}$ be a collection of closed subsets that is hereditary and additive. A countable union A of elements of $\mathscr{M}$ is called an *$\mathscr{M}$-absorber* if for every $D \in \mathscr{M}$ and every collection $\mathscr{U}$ of open subsets of X there is a homeomorphism h of X that is $\mathscr{U}$-close to the identity and that has the property $h(D \cap \bigcup \mathscr{U}) \subseteq A$. This definition is due to West [27].

This idea was extended as follows by Bestvina and Mogilski [8].

Let $\mathscr{K}$ be a class of spaces that is topological (i.e., homeomorphic images of elements of $\mathscr{K}$ belong to $\mathscr{K}$), additive, and hereditary with respect to closed sets. A subset X of a topological copy E of l^2 is a *generalized $\mathscr{K}$-absorber* if:

(1) $X = \bigcup_{i=1}^{\infty} X_i$, where each $X_i \in \mathscr{K}$ is a Z-set in X, and
(2) given an open cover $\mathscr{U}$ of X in E, a set $D \in \mathscr{K}$, a closed subset C of D, and a map $f : D \to \bigcup \mathscr{U}$ such that $f|C$ is a Z-embedding into X, there exists a Z-embedding $v : D \to X$ that is $\mathscr{U}$-close to f and that satisfies $v|C = f|C$.

A closed subset S of a space X is called a *Z-set* if continuous maps from X into $X \setminus S$ can be found arbitrarily close to the identity. A *Z-embedding* is an embedding whose range is a Z-set and a *σZ-set* is a countable union of Z-sets.

Bestvina and Mogilski [8] proved that any two generalized $\mathscr{K}$-absorbers in E are homeomorphic. We are especially interested in the case where $\mathscr{K}$ is the class $\mathscr{F}_{\sigma\delta}$ of all absolute $F_{\sigma\delta}$-spaces. Recall that the first Borel class consists of the absolute F_σ-spaces ($=$ σ-compacta) and the absolute G_δ-spaces ($=$ completely metrizable spaces). The second Borel class then consists of $\mathscr{F}_{\sigma\delta} = \{$ countable intersections of elements of $\mathscr{F}_\sigma\}$ and $\mathscr{G}_{\delta\sigma} = \{$ countable unions of elements of $\mathscr{G}_\delta\}$. In our case the space σ^ω is the standard generalized $\mathscr{F}_{\sigma\delta}$-absorber and we have:

Proposition 2.1. *A space X is homeomorphic to σ^ω if and only if* (a) *X is an absolute retract;* (b) *$X = \bigcup_{i=1}^{\infty} X_i$, where each $X_i \in \mathscr{F}_{\sigma\delta}$ is a Z-set in X;* (c) *X is homeomorphic to a space Y such that $X_f^\omega \subseteq Y \subseteq X^\omega$; and* (d) *$X$ contains a closed subset homeomorphic to X^ω.*

In this proposition X_f^ω stands for the subset of the product X^ω consisting of the sequences (x_i) that have almost all terms equal to some fixed point in X. Proposition 2.1 can be used to prove the following:

Theorem 2.2. *The following linear spaces are homeomorphic:*

(1) $\sigma^\omega = (l_f^2)^\omega$;
(2) $c_0 = \{(x_i) \in \mathbf{R}^\omega : x_i \to 0\}$ *endowed with the topology of coordinatewise convergence;*
(3) $\tilde{l}^p = \bigcap_{q>p} l^q$, $(0 \le p < \infty)$, *with the topology of coordinatewise convergence;*

(4) $\tilde{L}^p = \bigcap_{q<p} L^q$, $(0 < p \leq \infty)$, *with the topology of convergence in (Lebesgue) measure.*

Details will appear in [15].

3. Function spaces in the topology of pointwise convergence

Using a slight modification of Proposition 2.1 and some elements of the theory of Borel filters, we prove:

Theorem 3.1. *Let X be a nondiscrete, countable, completely regular space such that the function space $C_p(X)$ is an absolute $F_{\sigma\delta}$-set. Then $C_p(X)$ and $C_p^*(X)$ are homeomorpic to σ^ω.*

Here $C_p^*(X)$ is the subspace of $C_p(X)$ consisting of all bounded functions. Details will appear in [17]. Since for every countable metric space X the function space $C_p(X)$ is in $\mathscr{F}_{\sigma\delta}$, this theorem generalizes results of van Mill [22], Baars et al. [5], and Dobrowolski et al. [16]. As an application of Theorem 3.1 we can answer in the negative some questions of Arhangel'skiĭ [3, 4] by producing a countable, completely regular space X which fails to be a $b_{\mathbf{R}}$-space, a k-space, and an $\aleph_0$-space, while the function space $C_p(X)$ is homeomorphic to the C_p of the convergent sequence.

4. Classification of finite-dimensional pseudoboundaries and pseudointeriors

Let n and k be fixed integers such that $n \geq 1$ and $0 \leq k < n$. In addition, let $\mathscr{M}_k^n$ denote the collection of "tame" $\leq k$-dimensional compacta in $\mathbf{R}^n$. In [20] Geoghegan and Summerhill prove that there exists an $\mathscr{M}_k^n$-absorber. This set is called the k-dimensional universal pseudoboundary of $\mathbf{R}^n$, and we denote it by B_k^n. The k-dimensional universal pseudointerior s_k^n is the complement of B_{n-k-1}^n in $\mathbf{R}^n$. If $n \geq 2k+1$, then s_k^n can be seen as a k-dimensional analogue of Hilbert space in the topological category. These spaces are k-dimensional (locally) $(k-1)$-connected complete spaces which are universal for the k-dimensional spaces and which share the following properties with Hilbert space: (a) homogeneity, (b) every σZ-set is strongly negligible, and (c) Toruńczyk's discrete approximation property. We classify these spaces topologically by deriving the following:

Theorem 4.1. *B_k^n is homeomorphic to B_k^m if and only if s_k^n is homeomorphic to s_k^m if and only if $n = m$ or $n, m \geq 2k+1$.*

The following consequence is noteworthy. Consider the trefoil (or any other knot) in $\mathbf{R}^3$. Since the trefoil and the unknot are elements of $\mathscr{M}_1^3$, we may assume that they are subsets of s_1^3 or B_1^3. Since tame embeddings of S^1 are equivalent in $\mathbf{R}^4$, they are equivalent in s_1^4 and B_1^4 (see [19, Theorem 2.5] and [9, Theorem 1.2.13]). The theorem then implies that the trefoil is unknotted in both s_1^3 and B_1^3.

The method used in [13] to prove the theorem is strongly "infinite-dimensional" in spirit and is based on techniques similar to those that were used in §2, in [8], and in [24]. In order to show that B_k^m is homeomorphic to B_k^n for $m, n \geq 2k+1$, we use as a fixed model for these spaces a k-dimensional absorber B_k^ω in the Hilbert space l^2. The existence of B_k^ω is established in [10]. Let C stand for the cone of l^2 and let π be the projection $\mathbf{R}^n \times C \to \mathbf{R}^n$. We embed B_k^ω as a k-absorber in the topological Hilbert space $\mathbf{R}^n \times C$ in such a way that $\pi(B_k^\omega) = B_k^n$. Using a version of Bing's shrinking criterion that was developed by Toruńczyk [26] for incomplete spaces, we prove that $\pi | B_k^\omega : B_k^\omega \to B_k^n$ is a near homeomorphism if $n \geq 2k+1$.

The corresponding statement for s_k^n follows easily. The pseudo-interiors s_k^m and s_k^n contain embedded copies of B_k^m and B_k^n, respectively. By a classic Theorem of Lavrentiev the homeomorphism between B_k^m and B_k^n can be extended to a homeomorphism between G_δ-subsets X and Y of s_k^m and s_k^n, respectively. Since it can be shown that the complements of X and Y are strongly negligible, we have a homeomorphism between s_k^m and s_k^n.

5. Characterizing Hilbert space topology in terms of strong negligibility

In the late 1960s R. D. Anderson introduced the concept of a strongly negligible set to infinite-dimensional topology. He shows in [2] that in Hilbert space manifolds the strongly negligible sets are precisely the σZ-sets. Let us denote this property by SN= σZ. We investigate under what conditions the property characterizes the Hilbert space manifolds among the complete absolute neighbourhood retracts (ANRs). In [11] we proved the following:

Theorem 5.1. *A complete ANR is a Hilbert space manifold if and only if SN=* σ*Z and moreover if every compact subset is a strong Z-set.*

A closed subset S of a space X is called a *strong Z-set* if there exist continuous maps $f : X \to X$, arbitrarily close to the identity, such that the closure of $f(X)$ does not meet S (cf. the definition of Z-set in §2). This theorem is sharp in the sense that there exists an absolute retract, not homeomorphic to l^2, with SN= σZ and the property that compacta are Z-sets rather than strong Z-sets. This counterexample will be described in [12].

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