BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 22, Number 2, April 1990

MAPPING PROBLEMS IN COMPLEX ANALYSIS AND THE $\bar\partial$-PROBLEM

S. BELL

1. Introduction

Mark Twain's most famous short story, *The Celebrated Jumping Frog of Calaveras County*, was translated into many languages during his lifetime, including French. The French did not think the story was funny. Twain, in order to discover whether it was a flaw in the French persona or a flaw in the translation that rendered his hilarious story a flop, had the French translation translated word for word back into English. The French persona was exonerated; indeed the retranslation was not funny. In this paper, I will translate some results from one complex variable into the language of several variables, and then back again to one variable. The end result will differ from the original. I hope that the new perspective will enhance, rather than detract from, our understanding of the original.

Translating a result from one complex variable to several is more involved than merely saying, "Now let $n > 1$." Indeed, many arguments in one variable use the special relationship that exists between harmonic and holomorphic functions in the plane. In several variables, harmonic functions do not enjoy an elevated status; they are almost never mentioned. Thus, in several variables, a substitute must be found for the Laplace operator and the functions it annihilates. Generally, it is the $\bar\partial$-operator which replaces the Laplacian. In one variable, the $\bar\partial$-operator is given by $\partial/\partial\bar z = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. An excellent place to see classical one variable results proved using $\bar\partial$-techniques is in the first chapter of Hörmander's book on several complex variables [23]. For exam-

Received by the editors August 22, 1988.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 30C35, 32H99, 35N15.

Research supported by NSF grant DMS 8420754. This paper is based on an Invited Address given by the author at the AMS meeting in Indianapolis, April, 1986.

ple, there you will find the following three line proof of the Mittag-Leffler Theorem based on the fact that, given any function α which is C^∞ on a domain Ω, there exists a function $v \in C^\infty(\Omega)$ such that $\partial v/\partial \bar{z} = \alpha$. Suppose that a_i is a sequence of distinct points which does not have a limit point in the domain Ω and suppose that $P_i(z)$ is the principle part of a meromorphic function at a_i. Let D_i be a sequence of disjoint discs contained in Ω such that $a_i \in D_i$ and let ϕ_i be a function in $C_0^\infty(D_i)$ which is identically one on a small neighborhood of a_i. The function $u = \sum_{i=1}^\infty \phi_i P_i$ is in $C^\infty(\Omega - \{a_i\})$ and has the correct principle parts, but it is not holomorphic there. To get a holomorphic solution, we let v be a C^∞ solution to the $\bar{\partial}$-problem, $\partial v/\partial \bar{z} = \alpha$, where α is a C^∞ function and is equal to $\partial u/\partial \bar{z}$ on $\Omega - \{a_i\}$ and zero at each a_i. Now, $u - v$ is a solution to the Mittag-Leffler problem. (OK, four lines.) Besides harmonic functions, the cherished notion of conformality is also absent from the theory of functions of several complex variables.

Most of us agree that the theory of conformal mappings of planar domains is a sublimely beautiful subject. Since many of the most highly appreciated theorems of this subject have no obvious analogue in several complex variables, one might expect that the theory of holomorphic mappings between domains in $\mathbb{C}^n$ lacks the appeal of the classical one variable theory. I want to show how some of our favorite theorems in one complex variable can be viewed in order to obtain interesting results in several variables. In particular, I want to consider problems in several variables which spring from the Riemann Mapping Theorem in one variable.

I have tried to make this paper comprehensible to any reader who knows first year graduate level analysis.

2. The Riemann non-Mapping Theorem in several variables

(The prefix *non* belongs somewhere in the heading of this section, but it is difficult to decide where. I have found that putting *non* in front of the word *Theorem* yields a good title for a lecture on several complex variables.) Poincaré discovered that the unit polydisc and the unit ball in $\mathbb{C}^n$ are not biholomorphically equivalent, i.e., that there does not exist a one-to-one holomorphic map of one domain onto the other. Thus, the statement of a Riemann Mapping Theorem in several complex variables must be quite different than in one variable. This is one reason that I find mapping problems in two complex variables to be more than twice as interesting as those in one variable. (I must admit here that I have also heard it argued that this is why mapping problems in $\mathbb{C}^2$ are less than half as interesting as in $\mathbb{C}^1$.)

Let me begin by giving a proof of Poincaré's result which will introduce some of the features of holomorphic mappings in several variables that I will need later.

Theorem (2.1). *The ball and the polydisc in* $\mathbb{C}^n$ *are not biholomorphically equivalent.*

Proof. For $r \in \mathbb{R}$ and $a \in \mathbb{C}^n$, let $P(a; r)$ denote the polydisc $\{z \in \mathbb{C}^n : |z_i - a_i| < r ; i = 1, 2, \ldots, n\}$, and let $B(a; r)$ denote the ball $\{z \in \mathbb{C}^n : \sum_{i=1}^n |z_i - a_i|^2 < r^2\}$. Let us suppose that $f : P(0; 1) \to B(0; 1)$ is a biholomorphic mapping. We remark that it is a classical fact that the inverse of f is also a holomorphic map (see Rosay [38] or Krantz [28]). We may suppose that $f(0) = 0$. Indeed, because the cartesian product of Möbius transformations is a biholomorphic map of the polydisc onto itself, we may compose our given map with such a product map to obtain one that fixes the origin.

Let $u = \det[\partial f_i / \partial z_j]$ denote the holomorphic jacobian determinant of the mapping f. It can easily be shown by means of the Cauchy-Riemann equations that $|u|^2$ is equal to the classical *real* jacobian of f when viewed as a mapping from $\mathbb{R}^{2n}$ to itself (see [39, p. 11]). Furthermore, because the inverse of f is holomorphic, it follows that u cannot vanish on $P(0; 1)$. Let $F = f^{-1}$ and let $U = \det[\partial F_i / \partial z_j]$ denote the holomorphic jacobian determinant of F. Because $|u|^2$ is a *real* jacobian, it follows from the classical change of variables formula that

$$\int_{P(0;1)} |u(z)|^2 |\phi(f(z))|^2 \, dV_z = \int_{B(0;1)} |\phi(w)|^2 \, dV_w$$

where dV denotes Lebesgue measure on $\mathbb{R}^{2n}$. Thus, if $\phi \in L^2(B(0; 1))$, then $u(\phi \circ f) \in L^2(P(0; 1))$. (Here, the notation $u(\phi \circ f)$ stands for: *u times the quantity, ϕ composed with f.*) A similar application of the change of variables formula will yield the identity

$$\int_{P(0;1)} u(z)\phi(f(z))\,\overline{\psi(z)}\, dV_z = \int_{B(0;1)} \phi(w)\,\overline{U(w)\psi(F(w))}\, dV_w$$

which holds for all $\phi \in L^2(B(0;1))$ and all $\psi \in L^2(P(0;1))$. Indeed, to prove this formula, we may first assume that ϕ and ψ have compact support. The fact that $u(z)\,U(f(z)) = 1$ allows us to write

$$u(\phi \circ f)\bar{\psi} = |u|^2(\phi \circ f)\overline{((U(\psi \circ F)) \circ f)}.$$

Now it is clear that the identity follows from the classical change of variables formula. To obtain the more general result with ϕ and ψ in L^2, we apply a standard density and limit argument. Using $\langle\ ,\ \rangle_\Omega$ to denote the L^2 inner product on a domain Ω in $\mathbb{C}^n$, our formula can be abbreviated

$$\langle u(\phi \circ f)\,,\ \psi\rangle_{P(0;1)} = \langle \phi\,,\ U(\psi \circ F)\rangle_{B(0;1)}. \tag{2.1}$$

Now, because every holomorphic function on the polydisc has a power series expansion, and because the monomials z^α are orthogonal in L^2 on the polydisc, it follows that the set $\{z^\alpha : |\alpha| \geq 0\}$ forms an orthogonal basis for the space $H^2(P(0;1))$ of holomorphic functions which are in L^2 of the unit polydisc. The same reasoning applies to the ball. Thus, by expanding a function $h \in H^2(B(0;1))$ in its Taylor series, it can be seen that

$$\langle h\,,\ z^\alpha\rangle_{B(0;1)} = c_\alpha \frac{\partial^\alpha h}{\partial z^\alpha}(0)$$

for some constant c_α and for all holomorphic functions h in $L^2(B(0;1))$. (Note that in case $\alpha = (0, 0, \dots, 0)$, then $z^\alpha = 1$ and the formula becomes $\langle h\,,\ 1\rangle_{B(0;1)} = c_0 h(0)$.)

We shall now show that the mapping f must be linear. First, we will show that u is a constant. Indeed, using the conjugate of (2.1), we see that

$$\langle z^\alpha\,,\ u\rangle_{P(0;1)} = \langle U F^\alpha\,,\ 1\rangle_{B(0;1)} = c_0 U(0) F(0)^\alpha$$

and this last term is equal to zero if $|\alpha| > 0$ because $F(0) = 0$. Thus, the power series expansion for u must consist only of a single (nonzero) constant term. Now, to see that f is linear, observe that

$$\langle z^\alpha\,,\ u f_i\rangle_{P(0;1)} = \langle U F^\alpha\,,\ z_i\rangle_{B(0;1)} = c_i \frac{\partial}{\partial z_i}\{U F^\alpha\}(0)$$

and this last term is equal to zero if $|\alpha| > 1$. Thus, the power series expansion of $u f_i$ must be linear and, because u is a nonzero constant, we conclude that f is linear. It is now clear that no such f can exist. This finishes the proof of the theorem.

The proof given above is not Poincaré's original proof which involved an explicit computation of the dimensions of the automorphism groups of the ball and polydisc as Lie groups. The proof above has the virtue of generalizing to yield a proof of Cartan's lemma which states that if $f : \Omega_1 \to \Omega_2$ is a biholomorphic mapping between bounded circular domains in $\mathbb{C}^n$ which contain the origin and if $f(0) = 0$, then f must be linear. (A domain Ω is called *circular* if $z \in \Omega$ implies that $e^{i\theta} z \in \Omega$ for all real θ.)

3. Boundary behavior of biholomorphic mappings

The polydisc is topologically equivalent to the ball, yet the two domains are holomorphically inequivalent. It is reasonable to suspect that the problem lies with the fact that the ball has a C^∞ smooth boundary, whereas the polydisc has "corners" in its boundary. Indeed, it turns out to be a very good idea to consider properties of the boundaries when studying the problem of determining if two domains in $\mathbb{C}^n$ are biholomorphically equivalent; however, the relevant properties are more subtle than mere smoothness. The proof of Poincaré's Theorem given above, after some minor modifications, yields a proof that the ball $B(0; 1)$ and the *complex ellipsoid*, $E = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^4 < 1\}$ cannot be biholomorphically equivalent, because if they were, they would be so via a linear biholomorphic map, and this is clearly impossible. The ball and the complex ellipsoid both have C^∞ smooth boundaries. A key property that the boundaries of these two domains do not share concerns the degree to which they are *pseudoconvex*. The ball is the most basic example of a *strictly* pseudoconvex domain, and the ellipsoid is the simplest example of a *weakly* pseudoconvex domain. I shall not define pseudoconvexity here, but suffice it to say that it is a differential geometric property and that differential geometric properties of the boundaries of domains determine their holomorphic equivalence class rather than topological properties of their interiors. Indeed, Chern and Moser [15] built on the pioneering work of Poincaré and Cartan to produce a complete set of differential geometric boundary invariants which must be preserved under biholomorphic maps between smooth strictly pseudoconvex domains.

In order to see that the Chern-Moser invariants *are* preserved under a biholomorphic mapping, it is important to know that a biholomorphic map between smooth strictly pseudoconvex domains must extend smoothly to the boundary. C. Fefferman proved this result in [20]. I want to discuss here how the problem of proving that a holomorphic map exhibits good boundary behavior relates to some classical problems in one complex variable. I shall begin by giving a proof of a classical fact, first proved in 1887 by Painlevé [33], that the Riemann mapping function associated to a simply connected domain in the plane with C^∞ smooth boundary extends C^∞ smoothly up to the boundary. It is interesting to note that Painlevé proved his result about smooth extension long before Carathéodory proved his theorem about continuous extension (see [9] for the history of these theorems).

Theorem (3.1). *A biholomorphic map of a bounded simply connected planar domain with C^∞ smooth boundary onto the unit disc extends C^∞ smoothly up to the boundary.*

Proof. Suppose Ω is a bounded simply connected domain in the plane with C^∞ smooth boundary, and suppose f is a biholomorphic map of Ω onto the unit disc. It is easily seen that if $f(a)=0$, then $-\log|f(z)|$ is equal to the Green's function $G(z,a)$ associated to Ω for all $z\in\Omega$. Indeed, $G(z,a)$ is uniquely determined as the continuous function of z on $\bar{\Omega}-\{a\}$ which is zero on the boundary of Ω such that $G(z,a)+\log|z-a|$ extends to be harmonic as a function of z on Ω. If we differentiate the identity $-\log|f(z)|=G(z,a)$ with respect to z, we obtain

$$\frac{\partial}{\partial z}\left(\frac{1}{2}\log|f(z)|^2\right)=\frac{f'(z)}{2f(z)}=-\frac{\partial}{\partial z}G(z,a).$$

Thus, we obtain

$$f'(z)=-2f(z)\frac{\partial}{\partial z}G(z,a). \tag{3.1}$$

Now the classical elliptic theory for the Laplacian tells us that $G(z,a)$ is in $C^\infty(\bar{\Omega}-\{a\})$ as a function of z. To see this, note that $G(z,a)$ is equal to $-\log|z-a|$ minus the function $u(z)$ which is the solution to the boundary value problem: $\Delta u=0$ in Ω and $u(z)=-\log|z-a|$ on the boundary of Ω. Since $-\log|z-a|$ is C^∞ smooth on the boundary, the solution u to this problem extends C^∞ smoothly up to the boundary of Ω.

Carathéodory's theorem says that $f(z)$ is continuous up to the boundary of Ω. Hence (3.1) reveals that $f'(z)$ is continuous up to the boundary of Ω. Now it is a simple matter to repeatedly differentiate (3.1) with respect to z to see that all the derivatives of f extend continuously to the boundary. This completes the proof of the theorem. (Another nice proof of this theorem using classical potential theory can be found in Kerzman [24].)

One would like to adapt this simple proof to the several variable setting, however, two major obstacles present themselves. First, the Green's function associated to a domain in $\mathbb{C}^n$ bears no relation to holomorphic functions. In one variable, harmonic functions are locally the real part of holomorphic functions. In several variables, it is pluriharmonic functions which play this role and the Dirichlet problem for pluriharmonic functions is not well posed. Second, the ball in $\mathbb{C}^n$ does not qualify to be the general target domain in the theorem because of the failure of the Riemann Mapping Theorem in several variables. I shall now show how the proof of the one variable theorem above can be cleared of its dependence on the unit disc and the theory of harmonic functions and reconnected to the theory of the $\bar{\partial}$-problem so that a proof suitable for generalization to several variables is obtained.

Suppose that f is a Riemann mapping function associated to the simply connected planar domain Ω; that is, suppose that f is a biholomorphic map of Ω onto the unit disc. A classical formula from the theory of conformal mappings relates the Bergman kernel associated to Ω to the mapping f. If $f(a) = 0$ and $f'(a) > 0$, then we have the identity

$$f'(z) = cK(z, a)$$

where $c = \pi^{1/2}K(a, a)^{-1/2}$ and $K(z, w)$ denotes the Bergman kernel function of the domain Ω (see Bergman [11], Nehari [31]). Thus, we will be able to prove that $f(z)$ is C^∞ smooth up to the boundary if we can show that the Bergman kernel is C^∞ smooth up to the boundary. We will see that the Bergman kernel has the virtue of being directly linked to the $\bar{\partial}$-problem.

Theorem (3.2). *Suppose that Ω is a C^∞ smoothly bounded domain in the plane. The Bergman kernel function $K(z, w)$ associated to Ω is in $C^\infty(\bar{\Omega})$ as a function of z for each fixed $w \in \Omega$.*

Proof. In order to study the Bergman kernel function, we must first define the Bergman projection. Let $L^2(\Omega)$ denote the usual L^2 space on Ω with respect to Lebesgue measure on $\mathbb{R}^2$ equipped with the standard inner product $\langle\ ,\ \rangle_\Omega$. Because a function which is holomorphic on a closed disc assumes its average value at the center, it follows that if a sequence of holomorphic functions converges in $L^2(\Omega)$, then it converges uniformly on compact subsets of Ω. Thus, the space $H^2(\Omega)$ of holomorphic functions in $L^2(\Omega)$ is a closed subspace of $L^2(\Omega)$, and we can define the orthogonal projection P of $L^2(\Omega)$ onto $H^2(\Omega)$. This operator P is called the *Bergman projection.*

Let ϕ denote a real valued function in $C^\infty(\Omega)$ with compact support in Ω which is radially symmetric about a point $w \in \Omega$ such that $\int_\Omega \phi = 1$. It follows from the averaging property of holomorphic functions that

$$\langle h, \phi \rangle_\Omega = h(w)$$

for any holomorphic function h on Ω. If we project ϕ onto $H^2(\Omega)$, we obtain a holomorphic function $\kappa = P\phi$. This function κ has the property that

$$\langle h, \kappa \rangle_\Omega = \langle h, \phi \rangle_\Omega = h(w)$$

for all $h \in H^2(\Omega)$. The Bergman kernel $K(z, w)$ is defined by $K(z, w) = \kappa(z)$.

The Bergman projection is linked to the $\bar{\partial}$-problem via Spencer's formula

$$Pv = v - 4\frac{\partial}{\partial z} G \frac{\partial}{\partial \bar{z}} v \tag{3.2}$$

where G denotes the classical Green's operator which solves the Dirichlet problem: $\Delta(G\psi) = \psi$ with $G\psi$ equal to zero on the boundary of Ω. We shall prove Spencer's formula for functions v in $C^\infty(\bar{\Omega})$. To understand this formula, we need to study the operator $\Lambda\psi = 4\frac{\partial}{\partial z} G\psi$. We claim that if ψ is in $C^\infty(\bar{\Omega})$, then $u = \Lambda\psi$ is the unique solution to the $\bar{\partial}$-problem, $(\partial/\partial \bar{z})u = \psi$ with u orthogonal to $H^2(\Omega)$. Furthermore, $u \in C^\infty(\bar{\Omega})$. Indeed, it is a classical fact that G maps $C^\infty(\bar{\Omega})$ into itself; thus $u \in C^\infty(\bar{\Omega})$. It is also clear that $(\partial/\partial \bar{z})u = \psi$ because $4(\partial/\partial \bar{z})(\partial/\partial z) = \Delta$. Furthermore, because $G\psi = 0$ on the boundary of Ω, we may integrate by parts to obtain

$$\langle h, \Lambda\psi \rangle_\Omega = -\langle \partial h/\partial \bar{z}, 4G\psi \rangle_\Omega = 0$$

for any $h \in H^2(\Omega)$. Therefore, $u = \Lambda\psi$ solves the $\bar{\partial}$-problem. (To see that the solution is unique, observe that if u_1 and u_2 both solve the problem, then $u_1 - u_2$ is a function in $H^2(\Omega)$ which is orthogonal to $H^2(\Omega)$; this forces us to conclude that $u_1 - u_2 = 0$.)

We can now prove Spencer's formula. Assume that $v \in C^\infty(\bar{\Omega})$. Note that $v - \Lambda\frac{\partial v}{\partial \bar{z}}$ is a holomorphic function because it is annihilated by $\partial/\partial \bar{z}$. Furthermore, $\Lambda\frac{\partial v}{\partial \bar{z}}$ is orthogonal to $H^2(\Omega)$; hence $P\Lambda\frac{\partial v}{\partial \bar{z}} = 0$. We may now write

$$Pv = Pv - P\Lambda\frac{\partial v}{\partial \bar{z}} = P\left(v - \Lambda\frac{\partial v}{\partial \bar{z}}\right) = v - \Lambda\frac{\partial v}{\partial \bar{z}}$$

and Spencer's formula is proved.

Spencer's formula reveals that the Bergman projection maps $C^\infty(\bar{\Omega})$ into itself because G also enjoys this property. Now, because the Bergman kernel function is given as the projection of a function ϕ in $C^\infty(\bar{\Omega})$, we conclude that $K(z, w)$ is in $C^\infty(\bar{\Omega})$ as a function of z for each $w \in \Omega$. This completes the proof of the theorem.

Remark. The equipment that we used in the proof of Theorem (3.2) can be used to easily show that the Riemann map satisfies the identity, $f'(z) = cK(z, a)$ for some constant c. Indeed, if f maps Ω onto the unit disc D_1 with $f(a) = 0$, then the analogue of formula (2.1) is

$$\langle f'(\phi \circ f), \psi\rangle_\Omega = \langle \phi, F'(\psi \circ F)\rangle_{D_1}.$$

If we set $\phi = 1$, and if we let $\psi = h$, a function in $H^2(\Omega)$, then the conjugate of this identity yields

$$\langle h, f'\rangle_\Omega = \langle F'(h \circ F), 1\rangle_{D_1} = \pi F'(0)h(F(0)) = ch(a).$$

Thus, $f'(z)$ has the same effect when paired with a holomorphic function that $cK(z, a)$ does. Therefore, since f' is in $H^2(\Omega)$, the two must be equal.

I shall now show how the second proof of the smoothness of the Riemann mapping function can be adapted to several complex variables. No special knowledge of several complex variables will be required to understand the proof. In fact, because many people prefer the security of $\mathbb{C}^1$, and because the argument does not really use any special facts from several complex variables, I will set $n = 1$ at certain points in the proof of the following theorem to simplify the exposition. Before I state the theorem, let me remark that the Bergman projection associated to a domain Ω in $\mathbb{C}^n$ is defined exactly as in the one variable case as the orthogonal projection of $L^2(\Omega)$ onto the subspace $H^2(\Omega)$ of square integrable holomorphic functions.

Theorem (3.3). *Suppose* $f : \Omega_1 \to \Omega_2$ *is a biholomorphic mapping between bounded pseudoconvex domains in* $\mathbb{C}^n$ *with* C^∞ *smooth boundaries. If the Bergman projections associated to* Ω_1 *and* Ω_2 *preserve the space of functions which are* C^∞ *smooth up to the boundary, then* f *extends to be a* C^∞ *diffeomorphism of* $\bar{\Omega}_1$ *onto* $\bar{\Omega}_2$.

This extension of Fefferman's theorem was proved in [10]. The proof given here follows the method used in [6]. We shall let P_1 and P_2 denote the respective Bergman projections associated to Ω_1 and Ω_2. As in the proof of Poincaré's theorem, we shall let u denote the holomorphic jacobian determinant of f, F denote the inverse of f, and U denote the holomorphic jacobian determinant of F. In our present setting, formula (2.1) becomes

$$\langle u(\phi \circ f), \psi\rangle_{\Omega_1} = \langle \phi, U(\psi \circ F)\rangle_{\Omega_2} \tag{3.3}$$

which holds for all $\phi \in L^2(\Omega_2)$ and all $\psi \in L^2(\Omega_1)$. With this formula, we may now prove the *transformation formula for the Bergman projections under biholomorphic mappings:*

$$P_1(u(\phi \circ f)) = u\left((P_2\phi) \circ f\right).$$

Indeed, if ϕ is in $L^2(\Omega_2)$ and h is in $H^2(\Omega_1)$, then (3.3) yields

$$\begin{aligned}\langle u(\phi \circ f), h\rangle_{\Omega_1} &= \langle \phi, U(h \circ F)\rangle_{\Omega_2} \\ &= \langle P_2\phi, U(h \circ F)\rangle_{\Omega_2} = \langle u((P_2\phi) \circ f), h\rangle_{\Omega_1}.\end{aligned}$$

Here, we have used the fact that $U(h \circ F)$ is in $H^2(\Omega_2)$ which, recall, follows from the classical change of variable formula and the fact that $|U|^2$ is equal to the *real* jacobian determinant of F viewed as a mapping of $\mathbb{R}^{2n}$. Now, because $u(\phi \circ f)$ paired with an arbitrary function h in $H^2(\Omega_1)$ gives the same value as $u((P_2\phi) \circ f)$ paired with that function, we conclude that these two project to the same holomorphic function. But the second function is already holomorphic. Thus, the transformation formula is proved.

The next stage in the argument is to find a suitable replacement for the fact used in the one variable proof that the Bergman kernel is the projection of a C^∞ function with compact support. The following lemma will serve the purpose.

Lemma (3.4). *Suppose that* Ω *is a bounded domain in* $\mathbb{C}^n$ *with* C^∞ *smooth boundary. If* h *is a holomorphic function on* Ω *which is in* $C^\infty(\bar{\Omega})$, *then there is a function* ϕ *in* $C^\infty(\bar{\Omega})$ *which vanishes* *to infinite order at the boundary of* Ω *such the Bergman projection of* ϕ *is equal to* h.

This lemma does not seem to have been known in the one variable case. It can be used to simplify many classical one variable arguments about boundary regularity of holomorphic and harmonic functions.

Proof of the Lemma. Let us suppose, for the sake of illustration, that the boundary of Ω is real analytic and that h is holomorphic in a neighborhood of $\bar{\Omega}$. Then the Cauchy-Kovalevski theorem tells us that we can solve the Cauchy problem: $\Delta\psi = h$ with the boundary conditions, $\psi = 0$ and $\nabla\psi = 0$ on the boundary of Ω, where ψ is a real analytic function defined on a neighborhood of the boundary of Ω. Let χ be a C^∞ function on $\mathbb{C}^n$ which is equal to one on a neighborhood of the boundary of Ω and which has support which is compactly contained in the set where ψ is defined. I now claim that the function $\phi = h - \Delta(\chi\psi)$ is such that $P\phi = h$ and $\phi \in C_0^\infty(\Omega)$. That ϕ is in $C_0^\infty(\Omega)$ is clear. To see that $P\phi = h$, note that we can apply Green's identity to deduce that $\Delta(\chi\psi)$ is orthogonal to $H^2(\Omega)$. Indeed, because $\chi\psi$ and $\nabla(\chi\psi)$ vanish on the boundary, the boundary terms in Green's identity do not appear and we obtain

$$\int_\Omega \Delta(\chi\psi)\, \bar{g}\, dV = \int_\Omega (\chi\psi)\, \Delta\bar{g}\, dV = 0$$

if g is holomorphic because holomorphic functions are also harmonic. (I did say that harmonic functions were *almost* never mentioned in several complex variables.) Thus, $P(\Delta(\chi\psi)) = 0$ and we have that $h = Ph = Ph - P(\Delta(\chi\psi)) = P\phi$. The lemma is proved in case the boundary of Ω is real analytic and h is holomorphic on $\bar{\Omega}$.

To prove the lemma in the general case, we try to proceed in exactly the same way as above. We run into difficulty at the point where the Cauchy-Kovalevski theorem is invoked. At this point, we must use what I call "the C^∞ version of the Cauchy-Kovalevski theorem." It is possible to solve the Cauchy problem, $\Delta\psi = h$ with the boundary conditions $\psi = 0$ and $\nabla\psi = 0$ on the boundary of Ω, modulo functions which vanish to infinite order on the boundary. That is, there exists a function ψ in $C^\infty(\bar{\Omega})$ which satisfies the boundary conditions such that $h - \Delta\psi$ vanishes to infinite order on the boundary. Now it is clear that we may set $\phi = h - \Delta\psi$. The lemma is proved.

We may now use this lemma together with the transformation formula for the Bergman projections under biholomorphic maps to prove Theorem (3.3). Let h be a holomorphic function which is in $C^\infty(\bar{\Omega}_2)$ and let ϕ be a function in $C^\infty(\bar{\Omega}_2)$ which vanishes to infinite order on the boundary of Ω_2 such that $P_2\phi = h$. We now make the following claim.

Claim. *If* $\phi \in C^\infty(\bar{\Omega}_2)$ *vanishes to infinite order at the boundary, then* $u(\phi \circ f)$ *is in* $C^\infty(\bar{\Omega}_1)$.

Assuming the claim for the moment, let us finish the proof of the theorem. The transformation formula for the Bergman projections yields that

$$u(h \circ f) = u((P_2\phi) \circ f) = P_1(u(\phi \circ f)).$$

Since $u(\phi \circ f) \in C^\infty(\bar{\Omega}_1)$, and because P_1 preserves this class of functions, we deduce that $u(h \circ f) \in C^\infty(\bar{\Omega}_1)$. If we let $h = 1$, we see that $u \in C^\infty(\bar{\Omega}_1)$. If we let $h = z_i$, we see that $uf_i \in C^\infty(\bar{\Omega}_1)$. Thus, f extends C^∞ up to the boundary near boundary points where u does not vanish. But u cannot vanish on the boundary. Indeed, we may apply the same argument to the inverse mapping F to deduce that U extends smoothly to the boundary of Ω_2. Now $U(f(z)) = 1/u(z)$, and therefore, because U is bounded, u cannot vanish anywhere on $\bar{\Omega}_1$. This finishes the proof of the theorem.

Proof of the Claim. To convince you of the truth of the claim, let me assume temporarily that $n = 1$. Observe that the classical Cauchy estimates applied to a bounded function h on discs which are internally tangent to the boundary of Ω_2 yield that the derivatives of h satisfy an estimate of the form

$$\left|\frac{d^k h}{dz^k}\right| \leq C\, d_1(z)^{-k} \tag{3.4}$$

where $d_1(z)$ denotes the distance from z to the boundary of Ω_1 and C is a constant which is independent of z. We shall prove momentarily that f satisfies an estimate

$$d_2(f(z)) \leq c\, d_1(z) \tag{3.5}$$

where $d_2(z)$ denotes the distance from z to the boundary of Ω_2. Assuming this fact, we may finish the proof of the claim. Indeed, a derivative of $u(\phi \circ f)$ is a finite sum of terms of the form

$(D^\alpha u)((D^\beta \phi)\circ f)\prod_\gamma D^\gamma f$ where the D's stand for arbitrary real partial derivatives. Now, f satisfies an estimate of the form (3.4), and this implies that u satisfies $|d^k u/dz^k| \le C\, d_1(z)^{-k-1}$. Thus, the term $(D^\alpha u)\prod_\gamma D^\gamma f$ is controlled by a constant times $d_1(z)^{-m}$ where $m = |\alpha| + 1 + \sum_\gamma |\gamma|$. Since, given any positive integer m, the infinite order vanishing of ϕ makes it possible to find a constant k such that $|(D^\beta \phi)(z)| \le k\, d_2(z)^m$ for all z in Ω_2, we may use (3.5) to conclude that $|((D^\beta \phi)\circ f)|$ is less than a constant times $d_1(z)^m$. Thus, we may deduce that any derivative of $u(\phi\circ f)$ is bounded on Ω_1, and hence, that $u(\phi\circ f)$ is in $C^\infty(\bar{\Omega}_1)$.

To finish the proof of the claim, we must prove (3.5). Let λ be a solution to the Dirichlet problem: $\Delta\lambda = 1$ on Ω_1 and $\lambda = 0$ on the boundary of Ω_1. Note that λ is a subharmonic function on Ω_1 and the maximum principle for subharmonic functions implies that λ is negative on Ω_1. It therefore follows that $\lambda\circ F$ is a negative subharmonic on Ω_2. Furthermore, $\lambda\circ F$ extends to be continuous on $\bar{\Omega}_2$ and assumes the value zero on the boundary. Let R be chosen so that a disc of radius R may be rolled around the inside of the boundary of Ω_2 without ever touching more than one boundary point. Let $P(z, \zeta)$ denote the Poisson kernel for a disc of radius R which is internally tangent to the boundary of Ω_2 at a point p. To be precise, if we denote the center of this disc by W, then

$$P(z, \zeta) = \frac{R^2 - |z - W|^2}{2\pi R|\zeta - z|^2}.$$

Assume that ζ is on the circle S which is the boundary of this disc and that z lies along the inward pointing normal to the boundary of Ω_2 at p. Notice that

$$P(z, \zeta) \ge \frac{(R - |z - W|)(R + |z - W|)}{2\pi R(2R)^2} \ge C\, d_2(z)$$

where $C = 1/(8\pi R^2)$. We may now argue as in the proof of the classical Hopf Lemma to obtain that the positive superharmonic function $-\lambda\circ F$ satisfies the inequality

$$(-\lambda\circ F)(z) \ge \int_S P(z, \zeta)(-\lambda\circ F)(\zeta)\, d\sigma_\zeta \ge C\, d_2(z)\int_S -\lambda\circ F\, d\sigma$$

where $d\sigma$ denotes arc length on S. This inequality implies (3.5). Indeed, the last integral can be bounded from below by a positive constant which is independent of p. Furthermore, λ is C^∞ up to the boundary of Ω_2; thus, $-\lambda(w) \leq (constant)d_1(w)$. We therefore see that $d_1(F(z)) \geq (constant)(-\lambda(F(z))) \geq (constant)d_2(z)$ and (3.5) follows by replacing z by $f(z)$ in this inequality.

The claim is proved in case $n = 1$. The only place in the proof where a one variable argument was used was in the construction of the subharmonic function λ. In several variables, it is a special *plurisubharmonic* function that serves the purpose of the function λ; the possibility of constructing such a function is equivalent to pseudoconvexity [17]. Details and references can be found in [6].

Theorem (3.3) has been generalized to holomorphic mappings which are merely proper (see [8 and 18]). Also, see [18] for a history of the problem of proving boundary regularity of holomorphic mappings. Other important references on the subject include Webster [42] and Nirenberg, Webster, and Yang [32].

4. THE $\bar\partial$-PROBLEM IN SEVERAL COMPLEX VARIABLES

In order to deduce that the Bergman projection associated to a bounded domain in the plane with C^∞ smooth boundary preserves the space of functions which are C^∞ up to the boundary, we needed to know that the same property holds for the solution operator Λ to the $\bar\partial$-problem: $(\partial/\partial\bar z)(\Lambda\psi) = \psi$ with $\Lambda\psi \perp H^2$. In one variable, this follows directly from the fact that Λ may be expressed in terms of the classical Green's operator, $\Lambda = 4(\partial/\partial z)G$. In several variables, the problem is more difficult.

Suppose that Ω is a bounded domain in $\mathbb{C}^n$ with C^∞ smooth boundary. If v is a C^1 function on Ω, then $\bar\partial v$ is a differential one-form given by $\sum_{i=1}^n (\partial v/\partial\bar z_i)\, d\bar z_i$. To make this section more elementary, we shall think of $\bar\partial v$ as the n-tuple whose ith component is $\partial v/\partial \bar z_i$. Analogous to the situation in one variable, the Bergman projection associated to Ω may be written $Pv = v - \Lambda\bar\partial v$ where Λ denotes the solution operator to a $\bar\partial$-problem. However, this $\bar\partial$-problem takes the following more complicated form.

$\bar\partial$-Problem. *If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is an n-tuple of functions in $C^\infty(\bar\Omega)$ which satisfies the compatibility condition, $\partial\alpha_i/\partial\bar z_j = \partial\alpha_j/\partial\bar z_i$ for $i \neq j$, then $\Lambda\alpha$ is a function on Ω which solves the problem: $(\partial/\partial\bar z_i)(\Lambda\alpha) = \alpha_i$ for $i = 1, \ldots, n$ with $\Lambda\alpha \perp H^2(\Omega)$.*

It is a very difficult problem to determine when the operator Λ preserves the space $C^\infty(\bar\Omega)$. Indeed, in case Ω is not pseudoconvex, Λ does not even exist and it has been shown by Barrett [4] that the Bergman projection need not preserve the space $C^\infty(\bar{\Omega})$. In the strictly pseudoconvex case, however, it was proved by Kohn [25,26] that Λ does preserve $C^\infty(\bar{\Omega})$. He did this by relating Λ to an operator N, called the *$\bar{\partial}$-Neumann operator,* which is the analogue of the Green's operator in several variables. The best results in the weakly pseudoconvex case have been proved by Catlin [12,13,14] using the machinery set up by D'Angelo [16] to measure the degree of pseudoconvexity. I shall not discuss the problem further here; see [27] for an excellent survey.

5. Blaschke products in several complex variables

Like many topics in several variables that grew out of one variable results, the concept of a Blaschke product is interesting in many variables because it can readily be shown that the obvious generalization of the one variable notion does not make sense. (Recently, John D'Angelo has shown that Blaschke products in several variables do make sense provided that the dimensions of the target and the starting domain are allowed to be different.) In one variable, the set of Blaschke products is precisely the set of holomorphic mappings of the disc into itself which are *proper.* A map f is called *proper* if $f^{-1}(K)$ is compact whenever K is compact. Proper maps send sequences which tend to the boundary to similar sequences. What are the proper holomorphic self maps of the unit ball of $\mathbb{C}^n$ when $n > 1$? There are none which are not biholomorphic! This is Alexander's Theorem [1], which says figuratively that all Blaschke products in $\mathbb{C}^n$, $n > 1$, are Möbius transformations. I will now give a proof of Alexander's Theorem using some of the ideas developed in §3.

Theorem (5.1) (Alexander). *If f is a proper holomorphic self map of the unit ball in $\mathbb{C}^n$, $n > 1$, then f has a holomorphic inverse, i.e., f must be biholomorphic.*

Proof. A very elegant proof of this theorem was discovered by Rudin (see [39, p. 314]). I will give another proof which does not rely as heavily on special properties of the ball, and which therefore can be used to study proper mappings between more general domains.

Proper holomorphic mappings between domains in $\mathbb{C}^n$ behave similarly to their one dimensional counterparts. If $f : \Omega_1 \to \Omega_2$ is a proper holomorphic mapping between bounded domains in $\mathbb{C}^n$, then f maps Ω_1 *onto* Ω_2. Furthermore, there is a number m known as the *multiplicity* of the map, and there are sets V_1 and V_2 such that f is an m-sheeted covering map of $\Omega_1 - V_1$ onto $\Omega_2 - V_2$. The sets V_1 and V_2 are small; they are equal to the zero sets of holomorphic functions which do not vanish identically. (In one variable V_1 and V_2 are finite sets; in several variables, they are $n-1$ dimensional complex varieties.) These basic facts about proper holomorphic maps are explained very nicely in Chapter 15 of Rudin's book [39].

Our proof of Alexander's Theorem relies on the following transformation formula for the Bergman projections under proper holomorphic maps. This formula holds in one variable too and does not seem to have been known.

Lemma (5.2). *Suppose $f : \Omega_1 \to \Omega_2$ is a proper holomorphic mapping between bounded domains in $\mathbb{C}^n$, $n \geq 1$. Let u denote the holomorphic jacobian determinant of f and let P_1 and P_2 denote the respective Bergman projections associated to Ω_1 and Ω_2. Then*

$$P_1\left(u(\phi \circ f)\right) = u\left((P_2\phi) \circ f\right)$$

for all $\phi \in L^2(\Omega_2)$.

This formula is remarkable because it is exactly the same as the transformation formula for biholomorphic maps, yet the transformation formula for the Bergman kernels under a biholomorphic map does not have such a straightforward generalization to proper holomorphic maps.

Proof of the Lemma. I shall need the following fact which is an L^2 version of the Riemann Removable Singularity Theorem.

Fact. *If h is a holomorphic function on $\Omega_1 - V_1$ which is in $L^2(\Omega_1 - V_1)$ where V_1 is the zero set of a holomorphic function which is not identically zero, then h extends to be holomorphic on all of Ω_1.*

I shall prove this Fact assuming $n = 1$. The proof in several variables uses some elementary properties of complex varieties that I do not want to describe here. It is clear that the Fact is local. We may assume that Ω_1 is the unit disc and that h is holomorphic on the disc minus the origin. If $0 < \epsilon < 1$, let A_ϵ denote the annulus $\{z : \epsilon < |z| < 1\}$. Because every function which is holomorphic on A_ϵ can be expanded in a Laurent series, and because the functions $\{z^N : N \in \mathbb{Z}\}$ are orthogonal in $L^2(A_\epsilon)$, this set of functions forms an orthogonal basis for $H^2(A_\epsilon)$. Let $\| \ \|_\epsilon$ denote the norm in $L^2(A_\epsilon)$, and let the Laurent expansion of h be given by $\sum_{n=-\infty}^{\infty} a_n z^n$. Now $\|h\|_\epsilon = \sum_{n=-\infty}^{\infty} |a_n| \|z^n\|_\epsilon$, and as we let ϵ tend to zero, we see that a_n must be zero if $n < 0$ because the sum tends to a finite number as $\epsilon \to 0$, but $\|z^N\|_\epsilon \to \infty$ if $N < 0$. This finishes the proof of the Fact.

I can now prove the lemma. Since f is an m-sheeted cover of $\Omega_1 - V_1$ onto $\Omega_2 - V_2$, we may locally define m holomorphic mappings $F_1, \dots, F_m$ which map $\Omega_2 - V_2$ into $\Omega_1 - V_1$ and which are the local inverses to f. Let $U_1, \dots, U_m$ denote the holomorphic jacobian determinants of $F_1, \dots, F_m$, respectively. The analogue of formula (3.3) for a proper holomorphic mapping is

$$\langle u(\phi \circ f), \psi \rangle_{\Omega_1} = \left\langle \phi, \sum_{k=1}^{m} U_k(\psi \circ F_k) \right\rangle_{\Omega_2}. \tag{5.1}$$

To prove formula (5.1), we must first check that $u(\phi \circ f) \in L^2(\Omega_1)$ whenever $\phi \in L^2(\Omega_2)$ and that $\sum_{k=1}^{m} U_k(\psi \circ F_k) \in L^2(\Omega_2)$ whenever $\psi \in L^2(\Omega_1)$. First, notice that

$$\int_{\Omega_1 - V_1} |u|^2 |\phi \circ f|^2 \, dV = m \int_{\Omega_2 - V_2} |\phi|^2 \, dV$$

because f is an m-sheeted cover of $\Omega_1 - V_1$ onto $\Omega_2 - V_2$ and because $|u|^2$ is equal to the *real* jacobian determinant of f viewed as a mapping of $\mathbb{R}^{2n}$ to itself. But V_1 and V_2 are sets of measure zero. Thus $\|u(\phi \circ f)\|_{\Omega_1} = \sqrt{m} \, \|\phi\|_{\Omega_2}$. Notice that $\sum_{k=1}^{m} U_k(\psi \circ F_k)$, being a symmetric function, is well defined on $\Omega_2 - V_2$. Furthermore, since $|\sum_1^m a_i|^2 \le m \sum_1^m |a_i|^2$,

$$\int_{\Omega_2 - V_2} \left| \sum_{k=1}^{m} U_k(\psi \circ F_k) \right|^2 dV \le m \int_{\Omega_2 - V_2} \sum_{k=1}^{m} |U_k|^2 |(\psi \circ F_k)^2 \, dV$$

and this last integral is equal to

$$\int_{\Omega_1 - V_1} |\psi|^2 \, dV.$$

Thus $\| \sum_{k=1}^{m} U_k(\psi \circ F_k)\|_{\Omega_2} \le \sqrt{m} \, \|\psi\|_{\Omega_1}$. Now it is clear that formula (5.1) would be true if Ω_1 and Ω_2 were replaced by $\Omega_1 - V_1$ and $\Omega_2 - V_2$, respectively. But V_1 and V_2 are sets of measure zero. Therefore, (5.1) is proved.

We may now use (5.1) together with the Fact to prove the lemma. If $h \in H^2(\Omega_1)$, then $\sum_{k=1}^m U_k(h \circ F_k)$ is a holomorphic function on $\Omega_2 - V_2$ which is in $L^2(\Omega_2 - V_2)$. Thus, the Fact implies that $\sum_{k=1}^m U_k(h \circ F_k)$ extends past V_2 to be in $H^2(\Omega_2)$. If $\phi \in H^2(\Omega_2)$ and $h \in H^2(\Omega_1)$, then (5.1) yields that

$$\begin{aligned} \langle u(\phi \circ f), h\rangle_{\Omega_1} &= \left\langle \phi, \sum_{k=1}^m U_k(h \circ F_k)\right\rangle_{\Omega_2} \\ &= \left\langle P_2\phi, \sum_{k=1}^m U_k(h \circ F_k)\right\rangle_{\Omega_2} \\ &= \langle u((P_2\phi) \circ f), h\rangle_{\Omega_1}. \end{aligned}$$

Now, because $u(\phi \circ f)$ paired with an arbitrary function h in $H^2(\Omega_1)$ gives the same value as $u((P_2\phi) \circ f)$ paired with that function, we conclude that these two project to the same holomorphic function. But the second function is already holomorphic. Thus, the lemma is proved.

Assume that $f : B(0; 1) \to B(0; 1)$ is a proper holomorphic self map of the unit ball. We wish to show that f is biholomorphic. This will be accomplished if we prove that u, the jacobian determinant of f, does not vanish. (It then follows that f is a one-sheeted covering map of the ball onto itself and our result follows from basic topology.) The first step toward proving this will be to use Lemma (5.2) to show that f must extend holomorphically past the boundary of the ball.

Let P denote the Bergman projection associated to the ball. It is quite easy to show that the Bergman kernel function associated to the ball is given by $K(z, w) = c(1 - z \cdot \bar{w})^{-n-1}$ where $z \cdot \bar{w}$ stands for $\sum_{i=1}^n z_i\bar{w}_i$ and $1/c$ is equal to the volume of the ball as a set in $\mathbb{R}^{2n}$ (see Krantz [28, p. 50] or Stein [41, p. 21]). We know from the proof of Lemma (3.4) that for each monomial z^α, there exists a function ϕ_α in C^∞ of the unit ball *with compact support* such that $P\phi_\alpha = z^\alpha$. Now Lemma (5.2) reveals that

$$u(f^\alpha) = u\left((P\phi_\alpha) \circ f\right) = P\left(u(\phi_\alpha \circ f)\right).$$

This shows that $u(f^\alpha)$ is the projection of a function in $C_0^\infty(B(0; 1))$. The explicit formula for the Bergman kernel of the ball therefore implies that $u(f^\alpha)$ extends holomorphically past the boundary of the ball. This statement holds for each multi-index

α, including $\alpha = (0, 0, \ldots, 0)$. We now claim that this implies that f itself extends holomorphically past the boundary. To prove this, we will need to use the classical fact that the ring of germs of holomorphic functions at a point is a *unique factorization domain* (see Hörmander [23, p. 152] or Krantz [28, p. 242]). Let z_0 be a point in the boundary of the ball. By letting $\alpha = (0, \ldots, 0)$, we see that u extends to be holomorphic in a neighborhood of z_0. Furthermore, because u cannot vanish identically, it has a nontrivial factorization in the ring R of germs of holomorphic functions at z_0. Let

$$u = \prod_{i=1}^{r} W_i^{p_i}$$

denote the factorization of u in R. Let

$$uf_k = \prod_{j=1}^{s} V_j^{q_j}$$

denote the factorization of uf_k. Now, because $u(f_k)^m$ is in R for each m, we conclude that u^{m-1} divides $(uf_k)^m$ in R for each m. Thus $\{W_1, \ldots, W_r\}$ is a subset of $\{V_1, \ldots, V_s\}$. Let us renumber the V's so that u may be written

$$u = \prod_{i=1}^{r} V_i^{p_i}.$$

Because u^{m-1} divides $(uf_k)^m$ in R for each m, it also follows that $(m-1)p_i \leq mq_i$ for all m and for each i in the range $1 \leq i \leq r$. This forces us to conclude that $p_i \leq q_i$. Now we see that u divides uf_k in R and therefore, that f_k is an element of R, i.e., that f_k extends past the boundary of $B(0; 1)$ near z_0 as a holomorphic function. Thus, f extends holomorphically past the boundary of $B(0; 1)$.

Next, we must show that f preserves normal vectors on the boundary. To be more precise, we wish to show that if r is a function which is C^∞ on $\mathbb{C}^n$ such that $r = 0$ and $dr \neq 0$ on the boundary of $B(0; 1)$, then $d(r \circ f) \neq 0$ on the boundary of $B(0; 1)$. To see that this statement holds for all such r, it is enough to prove it for a single r because if r_1 and r_2 are two such functions, then $r_1 = \chi r_2$ for some C^∞ function χ which is nonzero in a neighborhood of the boundary of $B(0; 1)$. (This follows from the Implicit Function Theorem.) We shall prove the statement for $r(z) = |z|^2 - 1$ where we are using the standard notation $|z|^2 = \sum_1^n |z_i|^2$. It is clear that r satisfies the hypotheses. We must see that the normal derivative of $|f|^2 - 1$ does not vanish. To do this, note that $|f|^2 - 1$ is a subharmonic function on a neighborhood of the closed unit ball in $\mathbb{R}^{2n}$. Furthermore, $|f|^2 - 1$ assumes its maximum value of zero on the closed unit ball at every point of the boundary. Therefore, the classical Hopf lemma implies that the normal derivative of this function does not vanish. (The proof of the classical Hopf lemma is very similar to the proof we gave of inequality (3.5).)

We now know that $\rho(z) = |f|^2 - 1$ is what is called a *defining function* for the ball. This means that the ball is given as the set where ρ is less than zero, and that $d\rho \neq 0$ on the boundary. The ball is a geometrically convex domain; in fact it is a *strictly* convex domain. We wish to express convexity in terms of an analytic object in order to deduce some consequences about the boundary behavior of f. In this paragraph only, we shall use the convention that a subscript j stands for differentiation with respect to $\overline{z}_j$, and a subscript i stands for differentiation with respect to z_i. Thus, ρ_{ij} is shorthand for $\partial^2\rho/\partial z_i \partial \overline{z}_j$. Let H_ρ denote the *augmented hessian determinant* of ρ defined via

$$H_\rho = \det \begin{bmatrix} 0 & \rho_j \\ \rho_i & \rho_{ij} \end{bmatrix}$$

where the matrix inside the determinant is $(n+1) \times (n+1)$. It is a general fact that strict convexity of a domain defined by ρ implies that H_ρ does not vanish on the boundary. I shall prove this for the special case we are treating now. Indeed, if $r(z) = |z|^2 - 1$, an explicit computation reveals that H_r does not vanish on the boundary of the ball. Now, because r and ρ are both defining functions for the ball, there is a function χ which is non-zero in a neighborhood of the boundary of the ball such that $r = \chi\rho$. Another explicit computation shows that $H_r = \chi^{n+1} H_\rho$ on the boundary of the ball. Thus, H_ρ does not vanish on the boundary either. Let J_f denote the jacobian matrix of f. The chain rule yields that

$$\begin{bmatrix} 0 & \rho_j \\ \rho_i & \rho_{ij} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & J_f \end{bmatrix}^T \begin{bmatrix} 0 & r_j \\ r_i & r_{ij} \end{bmatrix} \circ f \begin{bmatrix} 1 & 0 \\ 0 & \overline{J_f} \end{bmatrix}.$$

Taking determinants of this identity gives

$$H_\rho = |\det J_f|^2 H_r \circ f.$$

Now, since H_ρ and H_r do not vanish on the boundary of the ball and since f maps the boundary into the boundary, we conclude that $u = \det J_f$ cannot vanish on the boundary. We can now use Hartog's theorem [23, p. 30] to see that u cannot vanish in the interior of the ball. Indeed, Hartog's theorem states that if g is holomorphic in a neighborhood of the the boundary of the ball, then g extends to be holomorphic on the whole ball (thus, there can be no isolated singularities for holomorphic functions of several variables). If we apply this theorem to $1/u$, we deduce that $1/u$ is holomorphic on the whole unit ball. This implies that u cannot vanish at any point in the ball. Thus, we have proved that f is an unbranched covering map of the ball onto itself, and therefore, that f must be a biholomorphic mapping. This completes the proof of the theorem.

I said at the beginning of the proof of Alexander's Theorem that the method of proof that I would use could be generalized. Let me explain briefly how this can be done. First, the ball can be replaced by any bounded domain with C^∞ smooth boundary. It can be shown [8,18] that a proper holomorphic self map of such a domain extends C^∞ smoothly up to the boundary. The function $r = |z|^2 - 1$ that we used in the proof must be replaced by what is called a *strictly plurisubharmonic* defining function. The Hopf lemma can be used to show that $r \circ f$ is a defining function for the domain. Finally, the consequences that we obtained using strict convexity are deduced from strict *pseudoconvexity*. On paper, the arguments appear exactly the same. The most general result about nonexistence of proper holomorphic self maps has been obtained by Pinčuk [34]. He proved that a proper holomorphic self mapping of any strictly pseudoconvex domain must be biholomorphic.

The survey article by Bedford [5] gives an excellent description of the leading problems in the study of proper holomorphic mappings between domains in $\mathbb{C}^n$.

6. Two positive Riemann Mapping Theorems in $\mathbb{C}^n$

Before I conclude this paper by listing some open questions, I must clear my conscience. I have probably convinced you that the notion of a Riemann Mapping Theorem in several variables is absurd. Let me show why that is not entirely true. In one variable, we know that a simply connected domain Ω that is not the whole plane is the same as the disc. It has a large group of biholomorphic self maps that correspond to the Möbius transformations of the disc. In fact, the automorphism group of Ω is transitive, i.e., for each pair of points z and w in Ω, there is a biholomorphic self map of Ω which maps z to w. This property characterizes those domains in the plane which are biholomorphic to the disc. Thus, a rather silly way to state the Riemann Mapping Theorem is: *A planar domain, not equal to $\mathbb{C}$, with a transitive automorphism group is biholomorphic to the disc.* The analogous statement for domains in $\mathbb{C}^n$ is not nearly so silly. Bun Wong [44] proved the next theorem in the strictly pseudoconvex case. Later, Rosay [37] was able to generalize it so that it can be stated as follows.

Theorem (6.1). *A bounded domain in $\mathbb{C}^n$ with C^2 smooth boundary that has a transitive automorphism group is biholomorphic to the unit ball in $\mathbb{C}^n$.*

Besides this theorem, there is one other theorem that has a claim to being called the *Riemann Mapping Theorem in $\mathbb{C}^n$*, namely Fridman's Theorem [21].

Theorem (6.2). *Suppose that D is diffeomorphic to the unit ball in $\mathbb{C}^n$. For any $\epsilon > 0$, there exist domains Ω_1 and Ω_2 contained in D and the unit ball, respectively, such that the boundary of Ω_1 is within a distance of ϵ from the boundary of D and such that the boundary of Ω_2 is within a distance of ϵ from the boundary of the unit ball, and such that Ω_1 and Ω_2 are biholomorphically equivalent.*

Actually, Fridman's Theorem is best described as an *approximate* Riemann Mapping Theorem.

There, I feel better now.

7. Some open questions

The unit polydisc in $\mathbb{C}^2$ has many proper holomorphic self mappings which are not biholomorphic. Indeed, the set of proper holomorphic self maps is precisely the set of mappings $f(z_1, z_2) = T(B_1(z_1), B_2(z_2))$ where B_1 and B_2 are finite Blaschke products and T is either the identity or the mapping which exchanges the first and second variables [40]. The polydisc does not have a smooth boundary. Pinčuk's Theorem [43] says that smooth strictly pseudoconvex domains have no nonbiholomorphic proper holomorphic self maps. Does there exist a smooth domain which has a nonbiholomorphic proper holomorphic self map?

The automorphism group of a domain is said to be *noncompact* if it is possible to find a point z_0 in the domain and a sequence of automorphisms Φ_j such that $\Phi_j(z_0)$ tends to the boundary as j tends to ∞. It is clear that a domain with a transitive automorphism group has a noncompact automorphism group. Greene and Krantz [22] realized that the only known domains with smooth boundaries in $\mathbb{C}^2$ which had noncompact automorphism groups were of the form $\{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^{2p} < 1\}$ where p is a positive integer. They conjectured that a smooth domain in $\mathbb{C}^2$ with noncompact automorphism group must be biholomorphically equivalent to one of these domains and they proved their conjecture in some special cases. Recently, Bedford and Pinčuk proved the Greene-Krantz conjecture for pseudoconvex domains with real analytic boundaries. The conjecture is wide open in the nonreal analytic case.

As we have seen, knowing that the boundary regularity of a solution to the $\bar\partial$-problem is as good as the regularity of the data has important applications to the problem of boundary regularity of holomorphic mappings. In this paper, I have discussed this idea in the C^∞ category. I would like to conclude this paper by mentioning some ideas I have had for studying similar questions in the real analytic category.

In one variable, we have the Schwarz Reflection Principle, which says the following. Suppose γ is a real analytic curve in the plane which passes through the origin and let D_ϵ denote a small disc about the origin such that $D_\epsilon - \gamma$ consists of exactly two connected components. If f is a mapping which is holomorphic on one side of γ in D_ϵ and continuous up to γ, and if the image under f of the curve γ is another real analytic curve, then f extends to be holomorphic past γ. The classical proof of the Schwarz Reflection Principle uses facts about harmonic functions, and therefore does not extend to several variables. We need to find a proof of the Schwarz Reflection Principle which uses the $\bar\partial$-problem. I shall now give a $\bar\partial$-proof of a weaker result to illustrate the possibilities.

Theorem. *Suppose that f is a holomorphic mapping on the upper half of the unit disc which extends to be C^1 up to the closure of the upper half disc. If the image of the real axis is the real axis, then f extends to be holomorphic past the real axis.*

This is a dumb theorem. I hope to give an interesting proof of a dumb theorem.

Proof. Let P denote the Bergman projection associated to the upper half disc D_+ and let z_0 be a point on the real axis with $-1 < \operatorname{Re} z_0 < 1$. Let ϕ denote a C^∞ function on $\mathbb{C}$ such that $\partial\phi/\partial z = 1$ near z_0 in $\mathbb{C}$ and such that $\phi = 0$ on the real axis. (The existence of ϕ follows from the Cauchy-Kovalevski theorem, or in the simple case at hand, we may take $\phi(z) = 2i \operatorname{Im} z$.) Let χ be a C^∞ function on $\mathbb{C}$ such that $\chi = 1$ in a neighborhood of z_0 and such that χ is compactly supported in the unit disc. The idea of the argument I am about to use stems from the proof I gave of the extendibility of a proper holomorphic self map of the unit ball in §5. Let $\psi = \partial(\chi\phi)/\partial z$. Note that ψ is equal to one in a neighborhood of z_0 in $\mathbb{C}$.

I want to show that f' is equal to the Bergman projection of a function which is zero in a neighborhood of z_0. This will imply that f extends holomorphically past z_0 via an argument using the Green's operator for the Laplacian. I claim that $f'(\psi \circ f)$ is orthogonal to holomorphic functions on the upper half disc. To see this, note that $f'(\psi \circ f) = \partial((\chi\phi) \circ f)/\partial z$, and therefore, integration by parts yields

$$\int_{D_+} f'(\psi \circ f)\, \bar{h}\, dV = -\int_{D_+} ((\chi\phi) \circ f)\, \frac{\partial \bar{h}}{\partial z}\, dV = 0$$

if h is in $H^2(D_+)$. Thus, $P(f'(\psi \circ f)) = 0$. Now we may write

$$f' = P\big(f'((1-\psi) \circ f)\big) = P\theta$$

where θ is function in $C(\bar{D}_+)$ which is zero in a neighborhood of z_0. Now the extendibility of f follows from the following property of the Bergman projection which I call *Condition* Q.

Condition Q. *If Ω is a bounded domain in the plane and if the boundary of Ω is a real analytic curve near a boundary point z_0, then $P\theta$ extends holomorphically past z_0 whenever θ is a function in $L^2(\Omega)$ which is supported away from z_0.*

This property is deduced from formula (3.2) and the classical fact that the Green's operator for the Laplacian is locally analytic hypoelliptic.

There are two questions that this proof raises. (1) How can the assumption about C^1 smoothness up to the curve be reduced to mere continuity up to the curve? (2) How can the argument be used in a several variable setting? In several variables, the C^1 assumption does not have such a bad ring to it. Recently, a great deal of progress has been made on generalizing the Schwarz reflection principle to several variables by Baouendi, Jacobowitz, and Treves (see [2, 3, 7, 19, 29, 35, 43]), but we suspect that much more general theorems can be proved.

If, after reading this paper, you want to learn more about several complex variables, I highly recommend the textbooks [23, 28, 30, 36, 39].

References

1. H. Alexander, *Proper holomorphic mappings in* $\mathbb{C}^n$, Indiana Math. J. **26** (1977), 137–146.
2. M. S. Baouendi, H. Jacobowitz, and F. Treves, *On the analyticity of CR mappings*, Ann. of Math. (2) **122** (1985), 365–400.
3. M. S. Baouendi and L. P. Rothschild, *Germs of CR maps between real analytic hypersurfaces*, Invent. Math. **93** (1988), 481–500.
4. D. Barrett, *Irregularity of the Bergman projection on a smooth bounded domain in* $\mathbb{C}^n$, Ann. of Math. (2) **119** (1984), 431–436.
5. E. Bedford, *Proper holomorphic mappings*, Bull. Amer. Math. Soc. (N.S.) **10** (1984), 157–175.
6. S. Bell, *Biholomorphic mappings and the* $\bar{\partial}$*-problem*, Ann. of Math. (2) **114** (1981), 103–113.
7. ——, *Analytic hypoellipticity of the* $\bar{\partial}$*-Neumann problem and extendability of holomorphic mappings*, Acta Math. **147** (1981), 109–116.
8. S. Bell and D. Catlin, *Boundary regularity of proper holomorphic mappings*, Duke Math. J. **49** (1982), 385–396.
9. S. Bell and S. Krantz, *Smoothness to the boundary of conformal maps*, Rocky Mountain J. Math. **17** (1987), 23–40.
10. S. Bell and E. Ligocka, *A simplification and extension of Fefferman's theorem on biholomorphic mappings*, Invent. Math. **57** (1980), 283–289.
11. S. Bergman, *The kernel function and conformal mapping*, Math. Surveys no. 5, Amer. Math. Soc., Providence, R. I., 1970.
12. D. Catlin, *Necessary conditions for subellipticity of the* $\bar{\partial}$*-Neumann problem*, Ann. of Math. (2) **117** (1983), 147–171.
13. ——, *Boundary invariants of pseudoconvex domains*, Ann. of Math. (2) **120** (1984), 529–586.
14. ——, *Subelliptic estimates for the* $\bar{\partial}$*-Neumann problem on pseudoconvex domains*, Ann. of Math. (2) **126** (1987), 131–191.
15. S. Chern and J. Moser, *Real analytic hypersurfaces in complex manifolds*, Acta Math. **133** (1974), 219–271.

16. J. D'Angelo, *Real hypersurfaces, orders of contact, and applications*, Ann. of Math. (2) **115** (1982), 615–637.

17. K. Diederich and J. E. Fornaess, *Pseudoconvex domains: Bounded strictly plurisubharmonic exhaustion functions*, Invent. Math. **39** (1977), 129–141.

18. ———, *Boundary regularity of proper holomorphic mappings*, Invent. Math. **67** (1982), 363–384.

19. ———, *Proper holomorphic mappings between real-analytic pseudoconvex domains in* $\mathbb{C}^n$, Math. Ann. **282** (1988), 681–700.

20. C. Fefferman, *The Bergman kernel and biholomorphic mappings of pseudoconvex domains*, Invent. Math. **26** (1974), 1–65.

21. B. Fridman, *An approximate Riemann mapping theorem in* $\mathbb{C}^n$, Math. Ann. **257** (1986), 49–55.

22. R. Greene and S. Krantz, *Biholomorphic self maps of domains*, Lecture Notes in Math., vol. 1276, Springer-Verlag, Berlin and New York, 1987, pp. 136–207.

23. L. Hörmander, *An introduction to complex analysis in several variables*, North-Holland, Amsterdam, 1973.

24. N. Kerzman, *A Monge-Ampère equation in complex analysis*, Proc. Sympos. Pure Math., vol. 30, Amer. Math. Soc., Providence, R.I., 1977, pp. 161–167.

25. J. J. Kohn, *Harmonic integrals on strongly pseudoconvex manifolds*. I, Ann. of Math. (2) **78** (1963), 112–148.

26. ———, *Harmonic integrals on strongly pseudoconvex manifolds*. II, Ann. of Math. (2) **79** (1964), 450–472.

27. ———, *A survey of the* $\bar{\partial}$*-Neumann problem*, Proc. Sympos. Pure Math., vol. 41, Amer. Math. Soc., Providence, R.I., 1984, pp. 137–145.

28. S. Krantz, *Function theory of several complex variables*, Wiley-Interscience, New York, 1982.

29. H. Lewy, *On the boundary behavior of holomorphic mappings*, Acad. Naz. Lincei **35** (1977), 1–8.

30. R. Narasimhan, *Several complex variables*, Chicago Lectures in Math., Chicago, Ill.-London, 1971.

31. Z. Nehari, *Conformal mapping*, Dover, New York, 1952.

32. L. Nirenberg, S. Webster, and P. Yang, *Local boundary regularity of holomorphic mappings*, Comm. Pure Appl. Math. **33** (1980), 306–338.

33. P. Painlevé, *Sur les lignes singulières des functions analytiques*, Thèse, Gauthier-Villars, Paris, 1887.

34. S. Pinčuk, *On proper holomorphic mappings of strictly pseudoconvex domains*, Siberian Math. J. **15** (1974), 909–917.

35. ———, *On the analytic continuation of holomorphic mappings*, Math. USSR-Sb. **27** (1975), 375–392.

36. R. M. Range, *Holomorphic functions and integral representations in several complex variables*, Springer-Verlag, Berlin and New York, 1986.

37. J.-P. Rosay, *Sur une caractérisation de la boule parmi les domains de* $\mathbb{C}^n$ *par son groupe d'automorphismes*, Ann. Inst. Fourier (Grenoble) **29** (1979), 91–97.

38. ———, *Injective holomorphic mappings*, Amer. Math. Monthly **89** (1982), 587–588.

39. W. Rudin, *Function theory in the unit ball of* $\mathbb{C}^n$, Springer-Verlag, Berlin and New York, 1980.

40. ———, *Function theory in polydiscs*, Benjamin, New York, 1969.

41. E. M. Stein, *Boundary behavior of holomorphic functions of several complex variables*, Math. Notes, no. 1, Princeton Univ. Press, Princeton, N.J., 1972.

42. S. Webster, *Biholomorphic mappings and the Bergman kernel off the diagonal*, Invent. Math. **51** (1979), 155–169.

43. ———, *On the reflection principle in several complex variables*, Proc. Amer. Math. Soc. **71** (1978), 26–28.

44. B. Wong, *Characterization of the ball in* $\mathbb{C}^n$ *by its automorphism group*, Invent. Math. **41** (1977), 253-257.

Department of Mathematics, Purdue University, West Lafayette, Indiana 47907