ON THE REGULARITY THEORY OF FULLY
NONLINEAR PARABOLIC EQUATIONS

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INTRODUCTION

Recently M. Crandall and P. L. Lions [3] developed a very successful method for proving the existence of solutions of nonlinear second-order partial differential equations. Their method, called the theory of viscosity solutions, also applies to fully nonlinear equations (in which even the second order derivatives can enter in nonlinear fashion). Solutions produced by the viscosity method are guaranteed to be continuous, but not necessarily smooth. Here we announce smoothness results for viscosity solutions. Our methods extend those of [1]. We obtain Krylov-Safonov (i.e. C^α estimates [8]), $C^{1,\alpha}$, Schauder $(C^{2,\alpha})$ and $W^{2,p}$ estimates for viscosity solutions of uniformly parabolic equations in general form. The results can be viewed as *a priori* estimates on the classical C^2 solutions. Our method produces, in particular, regularity results for a broad new array of nonlinear heat equations, including the Bellman equation [6]:

$$u_t - \sup_{\alpha \in A}[a_{ij}^\alpha(x,t)u_{ij} + b_i^\alpha(x,t)u_i + c^\alpha(x,t)u - g^\alpha(x,t)] = 0.$$

On the other hand, in the special case of linear equations, to which our method of course also applies, our proofs are much easier than the classical estimates for classical solutions, and also produce new results in this long-and well-studied field. For elliptic equations, similar results were obtained by Caffarelli [1], in the case that the equations do not involve the term Du.

We consider the following equation for a real-valued function u:

$$(1) \qquad u_t - F(D^2u, Du, u, x, t) = 0,$$

where $u_t = \partial u/\partial t$, $D^2u = (\partial^2 u/\partial x_i x_j)$, $Du = (\partial u/\partial x_i)$. Classically, there are two ways of attacking the problem of regularity

of solutions. One either writes down the solution explicitly, as in the theory of singular integrals, or differentiates the equation to get equations for the derivatives, as in [2], [4]. However, these methods do not apply to the very general equations we consider. The basic tools in our approach are the Aleksandrov-Bakel'man-Pucci-Krylov-Tso maximum principle [7, 12] and the method of compactness. Our method is fundamentally nonlinear, that is, it does not rely on a linearization of the equations. Loosely speaking, using the maximum principle, we can obtain tangent paraboloids for solutions (see Theorem 2 below), which then lead directly to second order derivative estimates. Theorem 2 is our main technical result; Theorems 3, 4 and 5 contain the main applications.

All the terminology used in this paper is in the sense of parabolic equations. See, for example, O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Ural'tseva [9]. Thus if the space variable x has homogeneity 1, then the time variable t has homogeneity 2; C^α means C^α in x and $C^{\alpha/2}$ in t; paraboloid means that it is quadratic in x and linear in t.

Definition. For a domain Q in R^{n+1} and $p > n + 1$, let $Y = W^{2,p}(Q)$, the Sobolev space of functions with the second order derivatives in L^p. By a standard estimate any function in Y is continuous. We say u is a *viscosity* solution of (1) provided that the following two conditions are satisfied:

  (a) if $u - \varphi$ attains a local maximum 0 at (x_0, t_0), then

(2) $\text{ess-inf-lim}[\varphi_t - F(D^2\varphi, D\varphi, \varphi, x, t)] \leq 0$

for all φ in Y.

  (b) if $u - \varphi$ attains a local minimum 0 at (x_0, t_0) in Q, then

(3) $\text{ess-sup-lim}[\varphi_t - F(D^2\varphi, D\varphi, \varphi, x, t)] \geq 0$

for all φ in Y.

  (2)(resp.(3)) is equivalent to requiring that (x_0, t_0) cannot be a density point of

   $\{\varphi_t - F(D^2\varphi, D\varphi, \varphi, x, t) > \delta(\text{resp.} < -\delta)\}$ for $\delta > 0$.

In a similar way, u in $C(Q)$ is said to be a *viscosity subsolution* (resp. *supersolution*) if (a)(resp.(b)) holds. We will write

        $u_t - F(D^2u, Du, u, x, t) \leq 0$ (resp. ≥ 0).

We say $F(M, P, v, x, t)$ is *uniformly elliptic* (i.e. (1) is *uniformly parabolic*) if there are two positive real numbers λ, Λ such that

(4) $\lambda|N| \leq F(M + N, P, v, x, t) - F(M, P, v, x, t) \leq \Lambda|N|$

where N is an arbitrary positive definite matrix and $|N|$ is its norm.

For simplicity, let us consider (1) in the form of

$$(5) \qquad u_t - F(D^2 u, Du, u, x, t) = g(x, t)$$

where F is such that $F(0, Du, u, x, t) = 0$. Here g is some bounded measurable function. Our methods, however, immediately extend to the more general case where f depends on u and Du.

For $r > 0$, define sets in $R^{n+1} = R^n \times R^1$ as follows:

$$Q_r = \{|x| < r\} \times (-r^2, 0], \qquad Q_r(x, t) = Q_r + (x, t);$$
$$\tilde{Q}_r = \{|x| < r\} \times (0, r^2], \qquad \tilde{Q}_r(x, t) = \tilde{Q}_r + (x, t).$$

For example, $Q_1 = B_1 \times (-1, 0]$.

Denote $\partial_p Q_1 = \partial B_1 \times (-1, 0] \cup B_1 \times \{-1\}$, the *parabolic boundary* of Q_1.

For a real number u, let u^+ and u^- be its positive part and negative part respectively. We have $u = u^+ - u^-$.

Our first result is the parabolic version of Aleksandrov-Bakel'man-Pucci-Krylov-Tso maximum principle [7, 12] for viscosity solutions. Let us introduce a special barrier function for u in $C(Q_1)$:

$$\Gamma(u) = \sup_{f \le -u^-} f(x, t)$$

where the supremum is taken over all the functions which are convex in x, decreasing in t and bounded above by $-u^-$.

Theorem 1. *Let u be a supersolution of (5) in Q_1. Assume $u \ge 0$ on $\partial_p Q_1$. Then*

$$(6) \qquad \sup(u^-) \le C \left(\int_{u = \Gamma(u)} |g^-|^{(n+1)} dx dt \right)^{1/(n+1)}$$

where C is a constant depending only on λ, Λ.

Let u be a supersolution as in Theorem 1, and let

$$(7)$$
$$G_h = \Big\{ (x_0, t_0) \in Q_1 : u(x, t) \ge u(x_0, t_0) + B(x_0, t_0)(x - x_0)$$
$$- \frac{A(x_0, t_0)}{2} |x - x_0|^2 + A(x_0, t_0)(t - t_0) \text{ for } (x, t) \in Q_1,$$
$$t \le t_0 \quad \text{with } u(x_0, t_0) + A + |B| \le h \text{ and } A \ge 0 \Big\}$$

i.e. G_h is the set where u has a tangent paraboloid with aperture h from below. Let

$$(8) \qquad\qquad B_h = Q_1 - G_h .$$

Theorem 2. *Let* u *be a positive supersolution in* $Q_2 \cup \tilde{Q}_2$ *with* $\inf_{\tilde{Q}_1} u \le 1$. *Then the measure of* B_h *satisfies*

$$(9) \qquad\qquad |B_h| \le C \frac{1}{h^\varepsilon} ,$$

for some $\varepsilon = \varepsilon(\lambda, \Lambda)$ *and* C *depending only on* $\|g\|_{n+1}$.

Remark. The proof of Theorem 2 is based on Theorem 1 and a Caldéron-Zygmund decomposition. From Theorem 2, we can get a *Harnack inequality* as in [10] and C^α *estimates* which give compactness for the set of solutions.

Moreover, if u is a solution of (5), then it has second order derivatives almost everywhere.

One immediate consequence of Theorem 2 is

$$(10) \qquad \int_{Q_1} |u_t|^{\varepsilon_0} + \int_{Q_1} |Du|^{\varepsilon_0} + \int_{Q_1} |D^2 u|^{\varepsilon_0} \le C$$

where $\varepsilon_0 = \varepsilon/2$, ε as in Theorem 2. Using (10), we have the following theorem:

Theorem 3. *Let* u *be a continuous* $(W^{2,p}$-*viscosity*) *solution of*

$$u_t - F(D^2 u, x, t) = g(x, t)$$

with $|u| < 1$ *and let*

$$S(x, t) = \sup_M \frac{|F(M, x, t) - F(M, 0, 0)|}{|M| + 1}$$

where the supremum is taken over the set of symmetric matrices. If the equation

$$v_t - F(D^2 v + M, 0, 0) = N$$

for M, N *on the surface* $F(M, 0, 0) = N$, *has interior* $C^{1,1}$ *estimates:*

$$(11) \qquad\qquad \|D^2 v\|_{L^\infty(Q_r)} \le \frac{C}{r^2} \|v\|_{L^\infty(Q_{2r})}$$

and

$$\|S(x, t)\|_{L^\infty(Q_1)} \le \delta_0(p, \lambda, \Lambda) ,$$

then for $p > n + 1$

$$(12) \qquad \int_{Q_{1/2}} |u_t|^p + |D^2 u|^p \leq C \left(\int_{Q_1} |g|^p + 1 \right).$$

We have the following lemma, whose proof relies on a new and useful general compactness argument as in the remark following Theorem 2. It plays a central role and is the real reason why we are able to develop a fully nonlinear method without linearization.

Approximation Lemma. *Let* u *be a solution of equation* (13) *in* Q_1 *with* $|u| \leq 1$.

$$(13) \qquad u_t - F(D^2 u, Du, x, t) = g(x, t).$$

Let

$$(14) \quad S_h(x, t) = \sup_{|q| \leq h} \frac{|F(M, p + q, x, t) - F(M, p, 0, 0)|}{|M| + 1}$$

where the supremum is taken over M, p, q.

Then, for any $\varepsilon > 0$, *there exist* $\delta(\varepsilon, \lambda, \Lambda)$ *and* $h(\varepsilon, \lambda, \Lambda)$ *such that if*

$$\begin{cases} v_t - F(D^2 v, 0, 0, 0) = 0, & \text{on } Q_{1/2} \\ v|_{\partial_p Q_{1/2}} = u, \end{cases}$$

then

$$\|u - v\|_{L^\infty(Q_{1/2})} \leq \varepsilon$$

provided that the small oscillation condition

$$(15) \qquad \|S_h\|_{L^{n+1}(Q_1)} + \|g\|_{L^{n+1}(Q_1)} \leq \delta(\varepsilon, \lambda, \Lambda)$$

for $h \geq h(\varepsilon, \lambda, \Lambda)$, *is satisfied.*

In addition to the compactness argument described above, the proof of this lemma depends on a uniqueness theorem of Jensen [5] and Theorem 3 for Pucci's maximum operators [11].

From this lemma, we derive $C^{1,\alpha}$, Schauder and $W^{2,p}$ estimates for general parabolic equations. For example, we have

Theorem 4. *Let* u *be a bounded solution of* (13).
  (a) *Suppose that the equation*

$$v_t - F(D^2 v, P, 0, 0) = 0$$

has interior $C^{1,\beta}$ *estimates (similar to* (11)).

Then there exists $\delta_0 = \delta_0(\beta, \lambda, \Lambda)$ *such that if*

$$(16) \qquad \varlimsup_{r \to 0} \left(\varlimsup_{h \to 0} \left(\fint_{Q_r} |s_h|^{n+1} \right)^{1/(n+1)} \right) + \|u\|_\infty \le \delta_0,$$

where $\fint_{Q_r} = 1/|Q_r| \int_{Q_r}$, *then* u *is* $C^{1,\alpha}$ *at* $(0, 0)$ *for* $\alpha < \beta$, *provided*

$$\left(\fint_{Q_r} |g|^{n+1} \right)^{1/(n+1)} \le Cr^{-1+\alpha}.$$

(b) *Suppose that the equation*

$$v_t - F(D^2 v, Dv + P, 0, 0) = 0$$

has interior $C^{1,\beta}$ *estimates.*

Then there exists $\delta_0 = \delta_0(\beta, \lambda, \Lambda)$ *such that if*

$$\varlimsup_{r \to 0} \left(\fint_{Q_r} |S_\infty|^{n+1} \right)^{1/(n+1)} \le \delta_0$$

then u *is* $C^{1,\alpha}$ *at* $(0,0)$ *for* $\alpha < \beta$, *provided*

$$\left(\fint_{Q_r} |g|^{n+1} \right)^{1/(n+1)} \le Cr^{-1+\alpha}.$$

Theorem 5. *Let* u *be a solution of*

$$u_t - F(D^2 u, x, t) = g(x, t).$$

If the equation

$$v_t - F(D^2 v + C, 0, 0) = D$$

for C, D *on the surface* $F(C, 0, 0) = D$, *has interior* $C^{2,\beta}$ *estimates (similar to* (11)) *and*

$$\left(\fint_{Q_r} |S|^{n+1} \right)^{1/(n+1)} \le Cr^\alpha,$$

(similar definition for S *as in Theorem 3) and*

$$\left(\fint_{Q_r} |g - g(0, 0)|^{n+1} \right)^{1/(n+1)} \le Cr^\alpha,$$

then u *is* $C^{2,\alpha}$ *at* $(0, 0)$ *for* $\alpha < \beta$.

Remarks 1. A combination of the above theorems gives a regularity theory for (1). We remark that the estimates in Theorems 3 and

5 hold for all equations of type (1) with natural growth condition as in [9].

2. All of the above theorems have analogs for boundary regularity of the Dirichlet boundary value problem.

3. All of the above theorems also have analogs for elliptic equations under similar conditions (change L^{n+1} to L^n).

Without the small oscillation condition (16), we have the following boundary regularity.

Theorem 6. *Suppose* $P_0 = (x_0, t_0) \in \partial_p Q_1$ *and* u *is a solution of* (5) *in* Q_1. *Also suppose that* $u|_{\partial_p Q_1} \equiv 0$. *Then for* $t_0 > -1$, u *is* $C^{1,\alpha}$ *at* P_0 *for some* $\alpha > 0$ *depending only on* λ, Λ *and* β, *provided*

$$\left(\fint_{Q_r(P_0)} |g|^{n+1} \right)^{1/(n+1)} \leq Cr^{\beta-1} .$$

For $t_0 = -1$, u *is* $C^{2,\alpha}$ *at* P_0 *for some* $\alpha > 0$, *provided*

$$\left(\fint_{\widetilde{Q}_r(P_0)} |g|^{n+1} \right)^{1/(n+1)} \leq Cr^{\beta} .$$

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