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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 20, Number 2, April 1989

0273-0979/89 $1.00 + $.25 per page

One-dimensional stable distributions, by V. M. Zolotarev. Translations of Mathematical Monographs, vol. 65, American Mathematical Society, Providence, 1986, ix + 284 pp., $92.00. ISBN 0-8218-4519-5

The concept of weak convergence of distribution functions plays a very fundamental role in Probability Theory. This concept may be defined as follows:

Let F be a distribution function, that is, a nondecreasing right-continuous function defined on $\mathbf{R}$ such that $F(-\infty) = 0$ and $F(+\infty) = 1$, and let $\{F_n : n \geq 1\}$ be a sequence of distribution functions. The sequence $\{F_n\}$ is said to *converge weakly* to the distribution function F on $\mathbf{R}$ if

$$F_n(x) \to F(x), \quad \text{as } n \to \infty$$

at all continuity points x of F. In this case we write $F_n \xrightarrow{w} F$, as $n \to \infty$. We note that the weak limit of the sequence $\{F_n\}$, if it exists, is unique. Moreover, let $\mathscr{C}_0 = \mathscr{C}_0(\mathbf{R})$ be the space of all bounded, real-valued continuous functions on $\mathbf{R}$. Let $\{F_n : n \geq 1\}$ be a sequence of distribution functions, and let F be a distribution function. Then $F_n \xrightarrow{w} F$ if and only if $\int_{-\infty}^{\infty} g\,dF_n \to \int_{-\infty}^{\infty} g\,dF$, as $n \to \infty$ for every $g \in \mathscr{C}_0$.

A necessary and sufficient condition for weak convergence was obtained by Lévy which can be stated as follows:

Let $\{F_n\}$ be a sequence of distribution functions and let $\{\varphi_n\}$ be the corresponding sequence of characteristic functions, that is, φ_n is the Fourier-Stieltjes transform of F_n for $n \geq 1$,

$$\varphi_n(t) = \int_{-\infty}^{\infty} e^{itx}\,dF_n(x), \qquad t \in \mathbf{R}.$$

Then $F_n \xrightarrow{w} F$ if and only if the sequence $\{\varphi_n\}$ converges (pointwise) on $\mathbf{R}$ to some function φ, which is continuous at the point $t = 0$. Moreover in this case, the limit function φ is the characteristic function of the limit distribution function F. This result plays a fundamental role in the study of the limit theorems for sums of independent (real valued) random variables.

The first result in this direction was obtained by Lévy which is known as the Lévy Central Limit Theorem and can be stated as follows.

Let $\{X_n : n \geq 1\}$ be a sequence of independently and identically distributed (i.i.d.) random variables with a finite variance $\sigma^2 > 0$. Let $S_n =$

$X_1 + X_2 + \cdots + X_n$, $n \geq 1$. *Then the sequence of distribution functions of the random variables* $(S_n - \mathscr{E}S_n)/\sigma\sqrt{n}$ *converge weakly to the distribution function of a Gaussian (or Normal) random variable with zero mean and unit variance, as* $n \to \infty$.

The book contains a systematic exposition of the analytic properties of one-dimensional stable distributions. It consists of an introduction and four chapters out of which the introduction and the second and third chapters contain the main bulk of information about the analytic properties of stable laws, while chapter 1 discusses examples of the occurrence of stable laws in applied problems and chapter 4 deals with the problem of statistical estimation of the parameters determining stable laws. The last chapter is intended to show the possibility of exploiting the analytic properties of stable laws in solving statistical problems. The structure of the book is such that only the introduction and the second and third chapters are interconnected while the first chapter is not necessary for understanding the remaining material and the fourth chapter makes only minimal use of the material in chapters 2 and 3. Information of a historical nature or concerning priority is reduced to a minimum in the main text and is discussed in the section entitled comments. The material in the introduction and the second and third chapters is fundamental to the book.

The material in the introduction is instrumental for the study of the subsequent chapters. In the theory of limit theorems for sums of independent real valued random variables Khintchine obtained the following basic result:

Let $\{X_{nj}\colon 1 \leq j \leq k_n,\ n \geq 1\}$ be a sequence of independent (in the individual series) random variables, and let

$$Z_n = X_{n1} + X_{n2} + \cdots + X_{nk_n} - A_n \tag{1}$$

where A_n's are some real numbers. Assume that the terms in the sum Z_n are uniformly infinitesimal, that is, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \max_{1\leq j\leq k_n} P(|X_{nj}| > \varepsilon) = 0. \tag{2}$$

Denote by F_{nj} and F_n the distribution functions of the random variables X_{nj} and Z_n respectively and by $\mathfrak{S}$ the set of all those distribution functions G that can be obtained as the weak limits of the functions F_n as $n \to \infty$. Then a distribution function G belongs to the set $\mathfrak{S}$ if and only if the characteristic function $\mathfrak{g}$ of G can be written in the form

$$\mathfrak{g}(t) = \exp\left\{ita - bt^2 + \int_{x\neq 0} (e^{itx} - 1 - it\sin x)\,dH(x)\right\} \tag{3}$$

where $a, b > 0$ are real numbers and the function H, which is defined on the whole x-axis except the point $x = 0$, is nondecreasing on both the semi-axis $x < 0$ and $x > 0$, tends to zero as $|x| \to \infty$ and satisfies the condition

$$\int_{0<|x|<1} x^2\,dH(x) < \infty. \tag{4}$$

Here G is called an infinitely divisible distribution, the function H is called the spectral function of G, and the representation (3) is called the canonical form of the characteristic function $\mathfrak{g}$.

The simplest variant of the scheme (1) is a sequence of linearly normalized sums of independently and identically distributed random variables of the form

$$(5) \qquad Z_n = (X_1 + X_2 + \cdots + X_n)B_n^{-1} - A_n$$

for $n \geq 1$, where $B_n > 0$ and A_n's are some real constants.

For condition (2) to hold in the scheme (5) it is sufficient that

$$(6) \qquad B_n \to \infty, \quad \text{as } n \to \infty.$$

It turns out that the property (6) holds whenever the distribution functions F_n of the random variables Z_n converge weakly to a nondegenerate distribution function G. A distribution function G is said to be *stable*, if it occurs in the scheme (5) as a weak limit of the sequence of distribution functions F_n, as $n \to \infty$. The set of all such functions G is called the family of stable laws and is denoted by $\mathscr{S}$. Clearly the family $\mathscr{S}$ of stable laws is a subset of the set $\mathscr{G}$ of all infinitely divisible laws. There are many different criteria for a distribution function to belong to the family $\mathscr{S}$, and any one of them can be taken as a definition of stable laws. One such criterion is as follows:

A distribution function G belongs to the family $\mathscr{S}$ if and only if G has the following property:

For any two positive real numbers b_1 and b_2, there exists a positive real number b and a real number a such that

$$(7) \qquad G(x/b_1) * (G(x/b_2) = G((x-a)/b).$$

Here $*$ denotes the operation of convolution in the theory of distributions. In this case the more restrictive relation

$$(8) \qquad G(x/b_1) * G(x/b_2) = G(x/b)$$

corresponds to a certain subfamily $\mathscr{M}$ of $\mathscr{S}$ which is called the family of *strictly stable* distribution laws. If in the scheme (5) we trace the situation leading to the limit distribution in the class $\mathscr{M}$, it turns out to correspond to the case when the linear normalization does not need to be centred, that is, when $A_n = 0$.

A description of the family $\mathscr{S}$ of stable laws and the subset $\mathscr{M}$ of it can be given as follows:

For each stable distribution G, the spectral function H corresponding to it has form

$$(9) \qquad H(x) = \begin{cases} -c_1 x^{-\alpha}, & \text{if } x > 0, \\ c_2(-x)^{-\alpha}, & \text{if } x < 0, \end{cases}$$

where c_1, c_2 and α are nonnegative real numbers and moreover $0 < \alpha < 2$.

A characterization of the family of $\mathscr{S}$ can be given as follows:

A nondegenerate distribution G belongs to the family $\mathscr{S}$, if and only if the logarithm of its characteristic function $\mathfrak{g}$ can be represented in the form

$$\text{(A)} \qquad \log \mathfrak{g}(t) = \lambda[it/\gamma - |t|^\alpha + it\omega_A(t, \alpha, \beta)]$$

where the real parameters $\alpha, \beta, \gamma, \lambda$ vary within the limits $0 < \alpha \leq 2$, $-1 \leq \beta \leq 1$, $-\infty < \gamma < \infty$, $0 < \lambda < \infty$ and

$$\omega_A(t, \alpha, \beta) = \begin{cases} |t|^{\alpha-1}\beta \tan\left(\frac{\pi\alpha}{2}\right), & \text{if } \alpha \neq 1, \\ -\beta(2/\pi)\log|t|, & \text{if } \alpha = 1. \end{cases}$$

Here the index A is used to distinguish the given form of expression for characteristic function of stable laws from the other forms of expression that are considered below. Another different form of expression for the characteristic function of stable laws is as follows:

The characteristic function $\mathfrak{g}$ of a nondegenerate distribution $G \in \mathscr{S}$ can be written in the form

$$\text{(B)} \qquad \log \mathfrak{g}(t) = \lambda[it/\gamma - |t|^\alpha \omega_B(t, \alpha, \beta)]$$

where

$$\omega_B(t, \alpha, \beta) = \begin{cases} \exp\left[-i\left(\frac{\pi}{2}\right)\beta K(\alpha)\operatorname{sgn} t\right], & \alpha \neq 1, \\ \frac{\pi}{2} + i\beta \log|t| \operatorname{sgn} t, & \text{if } \alpha = 1. \end{cases}$$

Here $K(\alpha) = \alpha - 1 + \operatorname{sgn}(1 - \alpha)$, and the parameters have the same domain of variation as in the form (A).

In what follows, the characteristic function $\mathfrak{g}$ will be accompanied by the parameter values such as $\mathfrak{g}(t) = \mathfrak{g}(t, \alpha, \beta, \gamma, \lambda)$. Using the estimate

$$|\mathfrak{g}_A(t, \alpha, \beta, \gamma, \lambda)| = \exp(-\lambda|t|^\alpha)$$

and the Fourier inversion formula, one can verify easily that the corresponding distribution function G has a density function g_A which exists and is uniformly bounded on the whole real axis, as is any derivative $g_A^{(n)}$ of g_A.

The distribution function and the density function of the stable laws with characteristic function $\mathfrak{g}(t, \alpha, \beta, \gamma, \lambda)$ will be denoted by $G(x, \alpha, \beta, \gamma, \lambda)$ and $g(x, \alpha, \beta, \gamma, \lambda)$ respectively and in shortened variants

$$\mathfrak{g}(t, \alpha, \beta) = \mathfrak{g}(t, \alpha, \beta, 0, 1)$$
$$G(x, \alpha, \beta) = G(x, \alpha, \beta, 0, 1)$$
$$g(x, \alpha, \beta) = g(x, \alpha, \beta, 0, 1).$$

In what follows the notation $Y(\alpha, \beta, \gamma, \lambda)$ and $Y(\alpha, \beta)$ are used for random variables with stable distribution $G(x, \alpha, \beta, \gamma, \lambda)$ and $G(x, \alpha, \beta)$ respectively.

One more rule, which is followed through the book, is stipulated as follows:

In all equalities connecting functions of random variables with the meaning that they have the same distribution (the symbol $\stackrel{d}{=}$ will be used for such equalities), the random variables on one side of an inequality (even when written the same) are understood as being independent. For example, with the use of this rule, (7) can be rewritten as follows:

A distribution function of the random variables $X_1 \overset{d}{=} X_2 \overset{d}{\neq} \text{const}$ belongs to the family $\mathscr{S}$ if and only if for any two positive real numbers b_1 and b_2, there exist a positive real number b and a real number a such that

$$b_1 X_1 + b_2 X_2 \overset{d}{=} bX_1 + a. \tag{10}$$

Here no special mention is made of the fact that X_1 and X_2 on the lefthand side of (10) are independent.

Finally a description of the class $\mathscr{M}$ of strictly stable laws is given as follows:

The distribution of the random variables $X_1 \overset{d}{=} X_2 \overset{d}{\neq} \text{const}$ belongs to $\mathscr{M}$ if and only if for any two positive real numbers b_1 and b_2, there exist a positive real number b such that

$$b_1 X_1 + b_2 X_2 \overset{d}{=} bX_1. \tag{11}$$

The characteristic function $\mathfrak{g}$ of the distributions in $\mathscr{M}$ have the representation

$$\log \mathfrak{g}(t) = -\lambda |t|^{\alpha} \exp\left[-i\left(\frac{\pi}{2}\right)\theta\alpha \operatorname{sgn} t\right] \tag{C}$$

where the parameters α, θ and λ vary within the limits

$$0 < \alpha < 2; \qquad |\theta| \leq \theta_2 = \min(1, 2/\alpha - 1): \qquad \lambda > 0.$$

The question that naturally arises is why such an abundance of forms for expressing the characteristic functions of stable laws exists. In studying the analytic properties of the distributions of stable laws we encounter groups of properties with their own diverse features. The expression of analytic relations connected with stable distributions can be simpler or more complicated depending on how advantageous the choice of the parameters determining the distributions for our problem turns out to be. By associating with a particular group of properties the parameterization form most natural for it, we minimize the complexity involved in expressing these properties. In this approach, the extraneous complexity is isolated from the problem and relegated to the formula for passing from one form of expressing the characteristic function $\mathfrak{g}$ to another.

The definition given above for the family $\mathscr{S}$ of one-dimensional stable distributions can be naturally extended to the case of finite-dimensional and even infinite-dimensional vector spaces. We consider a sequence of independently and identically distributed random variables, $X_1, X_2, \ldots,$ with values in k-dimensional Euclidean space $\mathbf{R}^k$ and form the sequence of sums

$$Z_n = \alpha_n(X_1 + X_2 + \cdots + X_n) - a_n, \qquad n \geq 1$$

normalized by some sequence of positive real numbers α_n and nonrandom elements $a_n \in \mathbf{R}^k$. The set $\mathscr{S}^k$ of all weak limits of the distribution of such sequences Z_n (as $n \to \infty$) is called the family of stable distributions in $\mathbf{R}^k$.

This is not the only way of generalizing the distributions in $\mathscr{S}$. If the sums $X_1 + X_2 + \cdots + X_n$ are normalized by nonsingular matrices α_n, then the concept of stable laws becomes essentially broader. At present very little is known about the properties of multidimensional stable distributions and in particular about their analytic properties. Neither the number nor the diversity of the facts known here can compare in any way with what is known about the one-dimensional stable distributions.

Chapter 2 makes a systematic study of the basic analytic properties of stable distributions belonging to the family $\mathscr{S}$. The analytic basis for the presentation of the material here is the explicit expression for the characteristic function of the distribution in the family $\mathscr{S}$ in one of the two forms (A) or (B). The stable law with the parameter values $\gamma = 0$ and $\lambda = 1$ in form (B) is called *standard*. The set of standard stable laws is denoted by $\mathscr{S}_0$. §2.1 deals with the elementary properties of stable laws in the family $\mathscr{S}$. The very simple expression for the characteristic functions of stable laws enables us at once to determine a whole series of interrelations between them. With these relations as a basis, it becomes possible, in particular, for us to reduce the study of analytic properties of the distributions in the family $\mathscr{S}$ to the study of the properties of distributions in various subfamilies of $\mathscr{S}$. §2.2 deals with the representation of stable laws by means of integrals. As already noted in the introduction, the function $|\mathfrak{g}(t)|$ is integrable on the whole real t-axis and hence the density function g of the standard distribution can be expressed by using the Fourier inversion formula

$$(12) \qquad g(x,\alpha,\beta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{itx}\mathfrak{g}(t,\alpha,\beta)dt = \frac{1}{\pi}\operatorname{Re}\int_0^{\infty} e^{itx}\mathfrak{g}(t,\alpha,\beta)dt.$$

Substituting in (12) the expression given in the introduction for the function $\mathfrak{g}(t,\alpha,\beta)$ in the form (B), we obtain

$$g(x,\alpha,\beta) = \frac{1}{\pi}\operatorname{Re}\int_0^{\infty} \exp\left\{-itx - t^{\alpha}\cdot\exp\left(-i\frac{\pi}{2}\beta K(\alpha)\right)\right\} dt$$

in case $\alpha \neq 1$ and

$$g(x,1,\beta) = \frac{1}{\pi}\operatorname{Re}\int_0^{\infty} \exp\left\{-itx - \frac{\pi}{2}t - i\beta t\log t\right\} dt$$

in case $\alpha = 1$.

The function $\mathfrak{g}$ can be written in a simple form in only four cases. That these stable laws actually have the density given below can be verified by direct computation of the corresponding integrals.

1. Lévy distribution

$$g(x,1/2,1) = \begin{cases} x^{-3/2}e^{-1/4x}/2\sqrt{\pi}, & \text{if } x > 0, \\ 0, & \text{if } x < 0. \end{cases}$$

2. Cauchy distribution

$$g(x,1,0) = 1/2(\pi^2/4 + x^2)^{-1}.$$

3. Gaussian distribution (with variance $\sigma^2 = 2$),

$$g(x,2,\beta) = e^{-x^2/4}/2\sqrt{\pi}.$$

4. The case obtained from the first by symmetric reflection

$$G(x, 1/2, -1) = g(-x, 1/2, 1).$$

For transformation of the integrals in (12) with the purpose of obtaining other expressions for the density function, it is essential to have an analytic continuation of the function $\mathfrak{g}(t, \alpha, \beta)$ in the complex z-plane with semi-axis $\operatorname{Re} z = t > 0$. This is carried out in the remaining part of this section. §2.3 investigates the duality law in the class $\mathscr{M}$ of strictly stable distributions, that is, it studies the relation connecting the distributions with the parameter $\alpha \geq 1$ and the distribution with parameter $\alpha' = 1/\alpha$ in the class $\mathscr{M}$. The duality property turns out to be a very convenient instrument for solving a number of problems such as, for instance, the problem of representing the density function of distributions in the class $\mathscr{S}$ by integrals, by convergent or asymptotic series, and so on. §2.4 deals with the analytic structure of stable distributions and their representation by convergent series. Stable distributions have density functions with uniformly bounded derivatives of all orders. Moreover the nth derivative of the density function of a standard stable distribution has the estimate

$$|g^{(n)}(x, \alpha, \beta)| \leq \frac{1}{\pi\alpha}\Gamma\left(\frac{n+1}{\alpha}\right)\left(\cos\left[\frac{\pi}{2}K(\alpha)\beta\right]\right)^{-(n+1)/\alpha}$$

This section is devoted to the study of the more suitable analytic structure of the stable distributions. §2.5 investigates the asymptotic expansion of stable distributions and §2.6 deals with the various integral transforms of stable laws. §2.7 discusses the problem of unimodality of stable distributions. A distribution function F is said to be *unimodal*, if there exists at least one value $x = a$ such that F is convex for $x < a$ and concave for $x > a$. In this case the distribution function F is said to have a mode at the point $x = a$. In this section it has been shown that every stable distribution is unimodal in the above sense. §2.8 expresses stable distributions as solutions of integral, integrodifferential and differential equations. §2.9 considers stable laws as functions of parameters. §2.10 studies denisty functions of stable laws as a class of special functions. Finally §2.11 introduces the concept of trans-stable functions and trans-stable distributions.

Chapter 3 studies special properties of strictly stable laws belonging to the class $\mathscr{M}$. The analytic properties considered in chapter 2 for the laws in the class $\mathscr{S}$ are based on the explicit expressions for the characteristic function of these laws. The fact that the characteristic functions are an analytic tool especially suited for working with sums of independent random variables is reflected in the material for chapter 2.

Mellin transforms and characteristic transforms of probability distributions serve as the analytic basis for the investigations carried out in chapter 3. These transforms fulfill the same purpose in the multiplicative scheme for independent random variables (M-scheme) as the characteristic functions have in the additive scheme (A-scheme). Therefore a large part of the material in this chapter involving various relations for distributions in the class $\mathscr{M}$ is presented in terms of products of independent random variables.

Chapter 4 is devoted to the study of estimations of the parameters of stable distributions. In this chapter a new approach to the problem of estimating the parameters of stable distributions is presented based on the use of explicit expressions for the corresponding characteristic transforms and the method of sample moments well known in statistics.

Chapter 1 gives examples of the occurrence of stable laws in applied fields. §1.1 studies a model of point source of influence. Examples include the gravitational field of stars; temperature distribution in a nuclear reactor; distribution of stresses in crystalline lattices; distribution of magnetic field generated by a network of elementary magnets. §§1.2 and 1.3 deal with the stable laws occuring in radio engineering and electronics and economics and biology respectively.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 20, Number 2, April 1989

0273-0979/89 $1.00 + $.25 per page

Nonlinear functional analysis, by Klaus Deimling. Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1985, xiv + 450 pp., $39.00. ISBN 3-540-13928-1

The author's purpose is to "present a survey of the main elementary ideas, concepts and methods which constitute nonlinear functional analysis." He has ably accomplished his goal. Moreover, by interlacing extensive commentary and foreshadowing subsequent developments within the formal scheme of statements and proofs, by the inclusion of many apt examples and by appending interesting and challenging exercises, Deimling has written a book which is eminently suitable as a text for a graduate course.

The central theme of nonlinear functional analysis is the study of systems of nonlinear equations—algebraic, functional, differential, integral—which are formulated as the solutions of an operator equation

$$F(x) = 0, \qquad x \in D, \tag{1}$$

where X and Y are linear spaces, $D \subset X$ and $F: D \to Y$ is some operator, which is not necessarily linear. There are, of course, many ways to cast the solutions of a system of equations as the zeros of a nonlinear operator, and the choice of set-up is often crucial. After the existence and multiplicity of solutions of (1) have been determined, more particular questions need to be considered. For instance, if a solution of equation (1) corresponds to a solution of a differential equation, one needs to study stability, regularity, dependence on initial data, etc.

In studying equation (1) there are analytical, topological and variational methods which can be invoked, as examples of which we mention iterative methods, degree theories and mini-max theorems, respectively. When $X = Y = \mathbf{R}^n$, the study of equation (1) for continuous F is already