The second point to observe is that, besides the left action on $\mathscr{E}(G/K)$ derived from left translations by G, there is a natural action of $\mathscr{E}'(G)$ on $\mathscr{E}(G/K)$ by convolution on the *right*. An elementary argument shows this action identifies $\mathscr{E}'(G)$ with the algebra of all intertwining operators for the left regular representation of G on $\mathscr{E}(G/K)$. Properties (iii) and (iv) result from this right action. But the right action and its intertwining property do not seem to be mentioned (except in the special case G compact, $K = \{1\}$, in §2 of Chapter 5, where it is conflated with regularity theorems (Theorems 2.3, 2.9, Lemma 2.4)). Even in the case of a compact symmetric space, the spectral projection property of the spherical function is not brought out; instead the Fourier expansion is effected as for an abstract representation, by convolving with characters (Theorem 4.3, formula (10), p. 538). This is despite the fact that in the example of spherical harmonics, in the Introduction, it is made quite clear (Proposition 3.2, p. 20, and Lemma 3.5, p. 22).

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Roger Howe
Yale University

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Global bifurcation and chaos: analytical methods, by Stephen Wiggins. Applied Mathematical Sciences, vol. 73, Springer-Verlag, New York, Berlin and Heidelberg, 1988, xiv + 494 pp., $49.95. ISBN 0-387-96775-3

Over the past twenty years, it has become abundantly clear that simple dynamical systems may behave in a very chaotic fashion. By now it is well known that simple (even quadratic) functions of a real variable may yield essentially random behavior when iterated, and that simple nonlinear ordinary differential equations in three dimensions (even with only one nonlinear term) may lead to solutions whose asymptotic behavior depends very sensitively on initial conditions.

This has come as quite a shock to applied scientists, but they have quickly realized the potential importance of this phenomenon—witness the number of conferences, books, and popular articles and television programs on the "new" science called chaos theory.

O. K. Dynamical systems may be chaotic. So what? If this were all there were to the subject, "chaos theory" would have emerged and then died rapidly in the mid-sixties when Smale first introduced his horseshoe example. This example showed that a simple diffeomorphism of the plane may possess spectacular chaotic behavior. But there is much, much more to the theory than the simple realization that solutions or orbits may behave chaotically. Indeed, as Smale showed in his original paper on the horseshoe map, the dynamics, despite their complexity, may be analyzed *completely*. In this sense, the word chaos is a misnomer: as Thurston has remarked, how can a system be called chaotic if you understand it completely?

Unfortunately, this aspect of chaos theory is not as well known in the applied sciences as it should be; indeed, many, though not all, of the recent contributions on the applied side of chaos theory are of the form "this system is chaotic, and so is that one, and that one, and that one," and so on.

Over the years there have been a number of important contributions in dynamical systems in which given chaotic systems have been thoroughly analyzed. There is the original Hadamard-Birkhoff-Morse-Hedlund analysis of geodesic flows on manifolds of negative curvature. This led directly to Smale's use of symbolic dynamics [S] to decipher the dynamics of the horseshoe map. Using this technique, Smale was able to prove that his simple map possessed a compact invariant in which

1. There were 2^n periodic points of period n.
2. Periodic points were dense.
3. There was also a dense, nonperiodic orbit.
4. Most other orbits diverged exponentially from each other.

These properties have since been found to occur in a wide variety of systems, namely Axiom A systems, and Smale's techniques can be extended to analyze these systems in a most satisfactory manner.

In a different direction, the branched manifold approach of R. F. Williams may be used to understand the dynamics on and near certain strange attractors. The most notable success of this approach is the Guckenheimer and Williams analysis of the Lorenz equations [G, W]. This system of ODEs is often heralded as the prototypical chaotic system, yet it too may be satisfactorily modeled and understood using the branched manifold approach. (To be precise, nobody knows if the actual Lorenz equations fit the Guckenheimer/Williams model, although this is universally believed to be true.)

One of the richest areas of inquiry in chaotic dynamics is the behavior of a dynamical system which features a homoclinic orbit. Such an orbit in the differential equations framework is one which is doubly asymptotic to an invariant set. Often, this invariant set is simply an equilibrium point or a closed orbit. Since the time Poincaré marveled at the complexity of a Hamiltonian system near such an orbit, mathematicians have attempted to understand the nearby behavior. Thanks to Smale's work, together with that of Silnikov and others in the Soviet Union, this behavior is now very well understood. The typical theorem asserts that, under generic hypotheses, the presence of a single homoclinic orbit implies the existence of a nearby invariant set on which the dynamics behave much like a variant of Smale's original horseshoe map.

There are many, many other instances of similar chaotic phenomena which may be nevertheless thoroughly analyzed. It goes without saying that this fact needs to be broadcast to the applied scientists who have recently discovered chaos.

In this regard, Stephen Wiggins' book *Global bifurcations and chaos: analytical methods* is a most welcome addition to the literature. The author's principal aim in the book is to introduce several analytical techniques which may be used by any scientist to prove the existence of chaos in his or her favorite differential equation.

After a brief discussion of the basics of dynamics, the author gives us his definition of chaos. One might quibble a bit with this definition, for chaos to Wiggins seems to be equivalent to the existence of a Smale horseshoe. There are many other manifestations of chaotic behavior which are not mentioned in the book as, for example, the hyperbolic toral automorphisms or strange attractors. These examples exhibit chaos on a much larger set than the simple set of measure zero which produces a horseshoe. However, the types of methods that Wiggins deals with in this book usually yield only a horseshoe or horseshoe-like dynamics, so he is perhaps right to confine his attention to this restricted notion of chaos.

The bulk of the book is devoted to homoclinic phenomena. Chapter three is devoted to the consequences of the existence of a single homoclinic orbit (to various invariant sets), and Chapter four is devoted to Melnikov's method. This is a global perturbation method which allows one to calculate explicitly the "generic" conditions on a homoclinic orbit necessary to produce chaos. These chapters are mathematically precise and heavily analytic.

I particularly like the final sections of the book, where Wiggins has applied all of the techniques developed previously to a variety of specific examples. The many figures in the book are in general exceptionally well drawn (the author thanks no less than 13 graphics artists for their help!). Some, however, are a little too complicated for human consumption—for example, Figure 2.3.9 contains at least 12 cones, 6 cylinders, 16 transverse sheets, and perhaps more, in a conglomeration which is difficult to sort out. A highlight of the book is the excellent set of references, including a good selection of Russian papers.

I doubt that the author intends his book to be used as a graduate level text in dynamics, although it grew out of a course he gave at Caltech. The book is too specialized for that. For example, it contains a treatment of the n-dimensional horseshoe map: most authors treat only the two dimensional case because it is technically more transparent. The book also presupposes a working knowledge of basic topics in dynamics such as sinks and sources, saddle nodes and period doublings, none of which are precisely defined. There is a 12 page refresher course on manifold theory, tangent bundles, manifolds with boundary and the like, and a 6 page section devoted to transversality, structural stability, and genericity, but I'm afraid that the reader will need to be previously exposed to these topics to fully appreciate them. One can learn the preliminary material elsewhere, as for example in the book of Guckenheimer and Holmes [GH]. Holmes was Wiggins' thesis advisor and, because of this, their books are naturally complementary and provide a good "one-two punch" in applied dynamics.

Wiggins' book is aimed primarily at the practicing applied scientist who has encountered chaos in his or her work. It will undoubtedly give these scientists an excellent bag of tricks necessary to recognize chaos and, more importantly, to analyze it. In this endeavor, the book succeeds admirably.

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Robert L. Devaney
Boston University

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Stochastic equations for complex systems, by A. V. Skorohod. Translated by L. F. Boron, D. Reidel Publishing Company, Dordrecht, 1988, xvii + 175 pp., $69.00. ISBN 90-277-2408-3

The modern theory of Markov processes in $\mathbf{R}^d$ developed out of the pioneering work of Kolmogorov, Feller, Lévy, Itô, and Dynkin, and many deep properties of the basic processes such as Brownian motion and Lévy processes have been investigated. Moveover based on the seminal work of Doob, the general theory of processes and stochastic calculus of semimartingales were systematically developed during the 1960s and 1970s (cf. Dellacherie and Meyer [**1**] for a complete exposition). On the other hand, the study of complex stochastic systems is still in its infancy. The book under review contains two chapters each devoted to an important aspect of this subject. The first chapter is devoted to the construction of continuous Markov processes in locally compact state spaces, including for example manifolds of variable dimension and manifolds with boundary.