are made about oscillatory and evolutionary spectra. The object is to see whether a local version of the type of spectral representation one has for a stationary process might hold for some nonstationary processes. Finally there are some words about harmonizable processes (a class of processes introduced by Loève) where a Fourier representation for the process is possible but not generally in terms of a random spectral function with orthogonal increments.

The book is extensively illustrated by many examples and illustrations. The second volume has over 800 references to an extensive literature in theory and applications with brief comments on the text in volume one or on the references. The work provides a much more rapid introduction to the probabilistic background, the extensive applications and basic results on stationary processes and spectral analysis than is possible in a conventional exposition and is excellent in this way. A reader who wants a more formal background should supplement the book by referring to other texts or to original papers. The two volumes are incredibly free of all except trivial typographical errors. The author is to be hailed for his extended and richly rewarding exposition.

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Representations of algebraic groups, by Jens Carsten Jantzen. Pure and Applied Mathematics vol. 131, Academic Press, Orlando, 1987, xiii + 443 pp., $59.50. ISBN 0-12-380245-8

The appearance of this book marks an important point in the development of the theory of rational representations of algebraic groups. Many different techniques have been introduced into this theory, especially during the last fifteen years. Jantzen's book gives the first comprehensive account of these new techniques, which often rely heavily on sophisticated algebraic geometry. Though firmly rooted in clear cut questions such as "What are the irreducible representations of $SL_n(p)$ in characteristic p?" the subject has become rather specialized. In order to explain how this has come about (and in an effort to avoid writing a review only for those whose time would be better spent browsing the book) I shall describe in some detail the basic framework of the theory. On grounds of space and ignorance, I shall concentrate on a few points. I apologize in advance to the many workers whose valuable contributions will be passed over in silence.

Throughout, G shall denote a semisimple algebraic group over an algebraically closed field K of characteristic $p > 0$. Such groups were classified by Chevalley in 1956/1958, **[1]**, and include the classical groups $SL_n(K)$, $Sp_n(K)$, $SO_n(K)$. Surprisingly, this classification is independent of p; it is described in terms of root systems in a manner similar to the classification of semisimple complex Lie algebras. In fact Chevalley has shown, **[2]**, that a semisimple group G over K may be constructed from a complex semisimple Lie algebra $\mathfrak{g}$ (having the same root system as G) by means of an integral lattice $\mathfrak{g}_{\mathbf{Z}}$ in $\mathfrak{g}$. One of the many benefits of the Chevalley construction is to give a way of defining G over $\mathbf{Z}$ (as a group scheme) and thereby opening the door to integral and modular (i.e. reduction mod p) techniques of representation theory. This is one of the main lines of current work and one to which we shall return.

A finite dimensional rational G-module (called simply a G-module in what follows) is a finite dimensional K-vector space V on which G acts in such a way that the representation $G \to GL(V)$ is a morphism of algebraic groups. The main focus of research, and the central problem in the area, is the determination of all irreducible G-modules. Despite a great deal of activity, the problem has been solved only in a few special cases (including $SL_2(K)$) mostly due to Brauer, Braden and Jantzen. However in general one does have a nice parametrization by dominant weights (due to Cartan-Chevalley) of the set of irreducible G-modules. This is achieved by induced modules and goes as follows. In G fix a maximal torus T (isomorphic to a direct product of r copies of K^*, the multiplicative group of the field K) and let Λ be the set of algebraic group homomorphisms $\lambda: T \to K^*$. Then Λ is an abelian group (called the lattice of integral weights) isomorphic to the free abelian group $\mathbf{Z}^r$ of r-tuples of integers. Let B be a Borel (i.e. maximal solvable) subgroup containing T (in the case $G = SL_n(K)$ one can take for T the diagonal matrices and for B the upper triangular matrices). Each integral weight λ extends uniquely to a representation $B \to K^*$ and so defines a one-dimensional B-module K_λ. We then have $\mathrm{Ind}_B^G K_\lambda$, the G-module induced (in the sense of algebraic group theory) from K_λ. Inside Λ is the set Λ^+ of dominant weights (corresponding to r-tuples of nonnegative integers, for G simply connected). For λ dominant, $\mathrm{Ind}_B^G K_\lambda$ has a unique irreducible submodule $L(\lambda)$ say. The modules $L(\lambda)$ (λ dominant) are pairwise nonisomorphic and each irreducible G-module is isomorphic to some $L(\lambda)$. Not only does this construction provide a natural labelling of the irreducible G-modules but it also provides a great deal of additional information.

There is a character theory of G-modules, which is analogous (and closely related to) the character theory of finite groups. A T-module V is the direct sum of its weight spaces $V^\lambda = \{v \in V : tv = \lambda(t)v$ for all $t \in T\}$, $\lambda \in \Lambda$. The formal character $\operatorname{ch} V$ of a T-module (or G-module) V records the weight space dimensions: precisely $\operatorname{ch} V = \sum_{\lambda\in\Lambda}(\dim V^\lambda)e^\lambda$, an element of the integral group ring $\mathbf{Z}\Lambda$ (with canonical basis $\{e^\lambda : \lambda \in \Lambda\}$). Two G-modules have the same composition factors (counting multiplicities) if and only if they have the same character. In the characteristic 0 case $L(\lambda) = \operatorname{Ind}_B^G K_\lambda$ (λ dominant) and $\operatorname{ch} L(\lambda)$ is given by the famous character formula of H. Weyl (originally proved for semisimple groups over $\mathbf{C}$ by using integration over a compact form). The central problem may therefore be interpreted as a search for a p-analogue of Weyl's character formula. A solution to this problem would give (via a theorem of R. Steinberg) the Brauer characters of the irreducible modules in the natural characteristic of the finite groups of Lie type as well as those for the symmetric groups in all characteristics (via a theorem of G. D. James).

An extra feature in characteristic p is the Frobenius morphism $\mathrm{Fr}: G \to G$. For a suitable realisation of G as a group of matrices this is given by raising the matrix entries to the pth power. Given a G-module V one thus gets a new G-module V^{Fr} by composing the representation $G \to GL(V)$ with the Frobenius morphism. Each dominant weight λ has a p-adic expansion $\lambda = \lambda_0 + p\lambda_1 + \cdots + p^j\lambda_j$ (we regard λ as an r-tuple of nonnegative integers and expand componentwise). Steinberg's tensor product theorem, [3], then asserts that $L(\lambda)$ is isomorphic to $L(\lambda_0)\otimes L(\lambda_1)^{\mathrm{Fr}}\otimes\cdots\otimes L(\lambda_j)^{\mathrm{Fr}^j}$. This reduces the problem to the determination of the $L(\lambda)$ where all the components of λ lie between 0 and $p-1$ (such λ are called restricted). In the case of $SL_2(K)$ the dominant wieghts correspond to nonnegative integers and $L(i)$ is $S^i(E)$, the ith symmetric power of the natural module E (of column vectors) for $0 \le i \le p-1$. In general one has no such pleasant description of the $L(\lambda)$ for λ restricted. However, there is a conjecture of Lisztig, [4], from which (if true) the characters of the $L(\lambda)$ may be computed, provided that p is large compared to the rank of G.

The most important new techniques are probably the infinitesimal method and the cohomology of line bundles on the space G/B. The infinitesimal method is to compare the representation theory of G with that of certain finite dimensional algebras $\mathfrak{u}_1 \subset \mathfrak{u}_2 \subset \cdots$ (introduced by J. E. Humphreys); $\mathfrak{u}_r$ is called the rth hyperalgebra of G and has dimension $p^{r\cdot\dim G}$. These algebras live in the dual $\operatorname{Hom}_K(K[G], K)$ of the coordinate ring and the union $\bigcup_{r=1}^\infty \mathfrak{u}_r$ has the same category of finite dimensional modules as G (the Verma Conjecture, now a theorem thanks to Sullivan and Cline-Parshall-Scott). One can then compare the representation theory of G (simple modules, principal indecomposable modules, cohomology...) with that of the algebras $\mathfrak{u}_r$. The introduction of the hyperalgebras has led to elegant new proofs of established results (e.g. Steinberg's tensor product theorem, [5]; Hilbert's 14th problem for reductive groups, [6], originally solved by Haboush, [7]). The algebras $\mathfrak{u}_r$ are in many ways similar to the group algebras $KG(p^r)$ (of the subgroups $G(p^r)$ of G of points defined over a finite field) but rather better behaved. In fact the $\mathfrak{u}_r$ behave like normal subgroups of G (this can be made more precise using the language of group schemes). One is thus in the unreasonably fortunate position of having all the tightness of structure implied by the (almost) simplicity of G and at the same time having a rich Clifford theory associated with infinitely many "normal" algebras $\mathfrak{u}_r$.

The cohomology of line bundles is a rather more sophisticated affair. The coset space G/B has the structure of a projective variety and every homomorphism $\lambda: T \to K^*$ gives rise to a line bundle $\mathscr{L}_\lambda$ on G/B. The induced module $\operatorname{Ind}_B^G K_\lambda$ is then the space of global sections $\Gamma(G/B, \mathscr{L}_\lambda)$ and, more generally, the derived functor modules $R^i\operatorname{Ind}_B^G K_\lambda$ (induction from B to G is only left exact) may be interpreted as the sheaf cohomology $H^i(G/B, \mathscr{L}_\lambda)$. One can then (as in the work of H. H. Andersen) prove results on $\operatorname{Ind}_B^G K_\lambda$ by proving them for all $H^i(G/B, \mathscr{L}_\lambda)$ (where certain inductive arguments may work better). This set up exhibits an interesting and complicated interplay between the representation theory of G and the geometry of G/B (and certain subvarieties called Schubert varieties). A key geometric result which has many representation-theoretic consequences is Kempf's vanishing theorem, which states that $H^i(G/B, \mathscr{L}_\lambda) = 0$ for all $i > 0$ if λ is dominant. (Kempf's original proof is entirely geometric but there are now *almost* entirely geometric proofs due to Andersen and Haboush.) To make matters still more interesting and complicated, Andersen has found it profitable in recent investigations to work over a principal ideal domain A instead of the field K. This has many advantages and allows one, by passing through the integers, to compare the theory at different characteristics (in the spirit of Brauer theory for finite groups). Probably the most important feature is the new light shed on the Weyl modules $V(\mu)$, μ dominant. By definition $V(\mu)$ is the dual module of $\operatorname{Ind}_B^G K_\lambda$, for $\lambda = -w_0\mu$ where w_0 is the longest element of the Weyl group. The character of $V(\mu)$ is given by Weyl's character formula. By working over a valuation ring and then using base change, Andersen produces a filtration $V(\mu) = V(\mu)^0 \supseteq V(\mu)^1 \supseteq \cdots$ (with $V(\mu)^i = 0$ for $i \gg 0$). The top quotient $V(\mu)^0/V(\mu)^1$ is $L(\mu)$ and there is an explicit formula for $\sum_{i>0} \operatorname{ch} V(\mu)^i$. This is known as Jantzen's sum formula (it was obtained in almost all cases by Jantzen by use of a contravariant form) and, as the author points out (p. 297) "This sum formula together with results in earlier chapters is, so far, the most efficient tool for calculating $\operatorname{ch} L(\mu)$."

There is, unfortunately, a high price to be paid for working over a principal ideal domain (or more general ground ring). In dealing with an algebraic group $\mathfrak{G}$ over $\mathbf{Z}$, say, we can no longer adopt the familiar and comforting convention of identifying $\mathfrak{G}$ with its set of points in some algebraically closed field but rather must take the Demazure-Gabriel approach that $\mathfrak{G}$ is a functor from commutative rings to groups. This scheme theoretic point of view must then be taken in connection with quotient spaces, representation theory, sheaf cohomology....

Part I of the book is devoted to the representation theory (and cohomology) of algebraic groups from the very general scheme-theoretic point of view indicated above, as well as groups over a general ground ring, this includes the representation theory of infinitesimal groups. The first part serves as a solid foundation for the representation theory of reductive groups (such as GL_n) given in Part II. The treatment in the first part is self contained, except that the reader is referred to Demazure-Gabriel, [**8**], for proofs at a couple of points and is expected to be familiar with sheaf cohomology as set up in Hartshorne, [**9**, Chapter III], say.

Part II is aimed at the representation theory of reductive groups over algebraically closed fields and in particular at the irreducible modules $L(\lambda)$. Among the many topics covered in the book and not mentioned above are: Jantzen's translation functors, cohomology of infinitesimal groups, and line bundles on Schubert varieties. The book is a systematic, concise and authorative treatment of the subject. Most of the material is not available elsewhere in book form (and some not in any form). The work is remarkably complete and up to date and has a comprehensive bibliography. This book will no doubt be used as a text for numerous study groups and will surely be the research worker's bible for mahy years to come.

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