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LAJOS TAKÁCS
CASE WESTERN RESERVE UNIVERSITY

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Combinatorics of finite sets by Ian Anderson, Clarendon Press, Oxford, 1987, xv + 250 pp., $45.00. ISBN 0-19-853367-5

Combinatorics has come of age. Just as most children pay attention primarily to their own interests, so the mathematical endeavor in the early years produces seemingly unrelated results and problem solutions. Puberty can be viewed as the beginning of awareness about the rest of the world, and in mathematics this brings survey articles that pull results together and place them in a common context. Finally, with maturity comes patience, yielding textbooks that enable the casual mathematician to acquire knowledge about a new area.

Ian Anderson's *Combinatorics of finite sets* brings to maturity a small but elegant area of combinatorics, sometimes called "extremal set theory." The primary object of study here is the *subset lattice*, which is the inclusion ordering on the subsets of an n-set. Note that the *elements* of the order are the *subsets* of the underlying set of elements. Extremal set theory focuses on purely combinatorial extremal problems, particularly those concerning chains and antichains. In an arbitrary partially ordered set (henceforth *poset*), a *chain* is a pairwise ordered collection of elements, and an *antichain* is a pairwise incomparable collection. Like other special posets, the subset lattice is a *ranked poset*, which means it has a well-defined rank function $r(x)$ such that $r(y) = r(x) + 1$ if $x < y$ and there is no element between them. We use N_k to denote the rank sizes, i.e., the number of $x \in P$ with $r(x) = k$.

Extremal set theory grew from two fundamental results about the problem of finding a maximum-sized antichain in a poset; this size is called the *width* of the poset. In 1928, Sperner determined the maximum-sized antichains in the subset lattice. If n is even, the unique maximum antichain consists of all subsets of size $n/2$, while if n is odd we may take all those of size $(n-1)/2$ or all those of size $(n+1)/2$, but no combination thereof. More generally, we say a ranked poset has the *Sperner property* if its width is $\max_k N_k$; i.e., some single rank forms a maximum-sized antichain. In 1950, Dilworth proved that the width of any finite poset equals the minimum number of chains needed to cover its elements. The relationship between these results is that the Sperner property can be verified by partitioning the poset elements into $\max_k N_k$ chains.

From the mid-1960s to mid-1970s, results poured forth about antichains and related families in the subset lattice and other special posets. Progress came from Erdös, Clements, Daykin, Greene, Harper, Hilton, Katona, Kleitman, and later Frankl, Griggs, and many others. Many proofs are now known for the theorems of Sperner and Dilworth, and the different approaches stimulated additional research, including variations of the Sperner property, generalizations to the *multiset lattice*, which is the same as the divisibility ordering on the divisors of an integer, and generalizations of these ideas to *k-families*, which are collections containing no chain of size $k+1$. Anderson calls these collections *k-unions*, since they are unions of k antichains.

The beauty of the results flowing from Sperner's Theorem is best displayed by *LYM orders*. Lubell gave a very short proof of Sperner's Theorem by a counting argument over all $n!$ maximal chains in the subset lattice. Each chain contains at most one element of an antichain, and an element of rank k lies in $k!(n-k)!$ of these chains. Thus, if an antichain contains a_k elements of rank k, we have $\sum a_k k!(n-k)! \leq n!$, or $\sum a_k/\binom{n}{k} \leq 1$. Yamamoto and Meshalkin obtained similar results, and a ranked poset P satisties the *LYM property* if the condition $\sum a_k/N_k \leq 1$ holds for any antichain in P. The LYM property clearly implies the Sperner property. One of Kleitman's major contributions was showing the equivalence of the LYM property to two other conditions. One is the *normalized matching property* of Graham and Harper, which requires for each k that $|\nabla F|/N_{k+1} \geq |F|/N_k$ whenever F is a subset of the kth rank and ∇F are those members of rank $k+1$ related to some member of F. The third condition is the existence of a list of maximal chains that for each k contain each member of rank k the same number of times, such as the list of all maximal chains in the subset lattice. Kleitman went on to show, for example, that every LYM order has the *strong Sperner property*, meaning that for any k the k largest ranks form a maximum-sized k-family. (Anderson does not use this term or *strict Sperner property*, which describes posets where all maximum antichains are single ranks.)

The major step of giving shape to the field and presenting it to the rest of the mathematical world was taken in the classic 1978 exposition by Curtis Greene and Daniel J. Kleitman, *Proof techniques in the theory of finite sets*, Studies in Combinatorics (MAA Studies in Mathematics vol. 17), edited by Gian-Carlo Rota. A later survey by West, *Extremal problems in partially ordered sets*, Ordered Sets (I. Rival, ed., Reidel (1982)), took a more encyclopedic approach, attempting to provide an annotated bibliography of what was then known. Konrad Engel and Hans-Dietrich O. F. Gronau provided an extremely thorough book-length treatment of much of this material, seeking full details and generality, in *Sperner theory in partially ordered sets* (Teubner-Texte zur Matematik, vol. 78, 1985). Also worth mentioning is *Combinatorics: set systems, hypergraphs, families of vectors, and combinatorial probability* (Cambridge Univ. Press, 1986), by Béla Bollobás. Bollobás's book has considerable intersection with Anderson's, and perhaps a similar intent. The viewpoint is different, interpreting set systems as hypergraphs rather than posets. Bollobás considers a wider range of questions, but is considerably more terse.

Until the publication of Anderson's book, the Greene-Kleitman article remained the physically and mathematically most accessible introduction to this engaging area. Anderson maintains the successful organization and approach taken by Greene and Kleitman, but the textbook format gives him room for both more details and more results. His presentation is patient but not meandering. The atmosphere is indicated by the focus on "Proof techniques" in the Greene-Kleitman title; more important than the most general results are the various techniques that can be used to prove the fundamental results and thereby yield extensions. This explains Anderson's inclusion of multiple proofs of fundamental results, often in different chapters.

Since the Greene-Kleitman paper has represented this field for so long, let us consider some ways in which Anderson enlarges upon it. He draws on the results and new proofs of the past decade that have helped to round out the field. LYM orders have become so fundamental that discussion of them is spread throughout the book. Concerning the Sperner property for special posets, Anderson outlines Shearer's proof of Canfield's result that for large n the lattice of partitions of an n-set is *not* Sperner, contrary to a conjecture that stood for ten years.

For posets whose rank sizes N_k form a symmetric and unimodal sequence, the Sperner property is implied by the existence of symmetric chain decompositions. A *symmetric chain decomposition* of a poset is a partition of it into chains that hit consecutive ranks and are symmetric around the middle rank. By Dilworth's Theorem, the middle rank is thus a maximum antichain, and in fact a symmetric chain decomposition implies the strong Sperner property. Anderson's discussion of symmetric chain decompositions and their applications is fairly thorough. There are a large number of special posets for which the existence of symmetric chain decompositions is an open question. Anderson mentions the most important: $L(m,n)$ has several descriptions, one being the poset of nonnegative integer sequences $0 \leq a_1 \leq \cdots \leq a_m \leq n$, ordered by $\overline{a} \leq \overline{b}$ if $a_i \leq b_i$ for all i. Anderson includes an outline of Lindström's proof that $L(3,n)$ has such a decomposition; West and Riess independently proved it for $L(4,n)$. Stanley has used powerful and difficult results in algebraic geometry to show that various posets, including $L(m,n)$, are Sperner. These methods do not yield symmetric chain decompositions. Since they are algebraic rather than combinatorial, they are beyond the scope of this book, and Anderson mentions but does not explore them.

The discussion of correlational inequalities is also thorough and particularly welcome, since many important results about these have appeared since the Greene-Kleitman paper. Given two unrelated elements x, y in a poset P, the *probability* of $x < y$ is defined to be the fraction of the linear orders consistent with P in which $x < y$. The outstanding result in this area is the XYZ Inequality, which states that in any P the events $x < y$ and $x < z$ are positively correlated, i.e., knowing $x < z$ makes it more likely that $x < y$. Anderson gives a complete proof of this, starting with the very powerful Ahlswede-Daykin Inequality (for subsets). Given a function f on a poset, let $f(X)$ for a collection X denote $\sum_{x \in X} f(x)$, let $X \vee Y = \{x \cup y \colon x \in X,\ y \in Y\}$, and let $X \wedge Y = \{x \cap y \colon x \in X,\ y \in Y\}$. The Ahlswede-Daykin Inequality, called the *Theorem of the four functions* by Bollobás, says that if α, β, γ, δ are four nonnegative functions on the subset lattice satisfying $\alpha(x)\beta(y) \leq \gamma(x \cup y)$ $\delta(x \cap y)$ for all subsets x, y, then $\alpha(X)\beta(Y) \leq \gamma(X \vee Y)\delta(X \wedge Y)$ for all collections X, Y of subsets. Its many applications include the FKG Inequality, a poset generalization of Chebyshev's Inequality originally discovered in statistical mechanics. Shepp used the FKG Inequality to prove the XYZ Inequality.

One of the strongest statements about the order relation on subsets is the Kruskal-Katona Theorem; Anderson presents this and its extension to multisets by Clements and Lindström. The chapters concerning these and their applications are harder than the earlier ones. The Kruskal-Katona Theorem answers the question of how to choose m k-element subsets of a set to minimize the number of $k-1$-element subsets contained in one or more of them. The answer is to take the first m k-element subsets in the lexicographic ordering of their representation as binary vectors. Extremal problems like these ultimately came from coding theory. Among the applications of the Kruskal-Katona Theorem is the Erdös-Ko-Rado Theorem, which itself has many interesting proofs and extensions. The EKR Theorem states that for $k \leq n/2$ the maximum-sized collection of pairwise-intersecting k-subsets of a n-set consists of all the k-sets containing one particular element.

Finally, Anderson closes the book with the important Greene-Kleitman extension of Dilworth's Theorem to k-families in arbitrary posets. Given any chain partition and any (maximum) k-family, each s-element chain in the partition contributes to the k-family no more than $\min\{k,s\}$ elements. (This proves that a symmetric chain decomposition implies the strong Sperner property.) Greene and Kleitman proved that for any k and any poset there exists a chain partition for which equality holds; this is called a *k-saturated* chain partition. In fact, they proved that for each k there exists a partition that is both k- and $k+1$-saturated. The original proof was quite long. Anderson presents the shorter recent proof; Saks showed that k-saturated partitions exist, and then Perfect used this to get the simultaneous k, $k+1$ property.

The variety of proof techniques available in this subject and Anderson's focus on these techniques make his book an excellent text for a topics course in discrete mathematics. I ran such a course in 1980 based on the Greene-Kleitman article. After presenting the fundamental material, it was necessary to direct the students to the literature to cover material for which Greene and Kleitman did not have room. Anderson has now done this job for us, making the material easy to teach. The typography is very attractive, and the proofs are thorough and readable. Also important are the 156 exercises, with hints or solutions at the back (this is perhaps better for the mathematician interested in a new area than it is for teaching). Most of the exercises seem relatively easy. They incorporate many of the particular results in the literature that do not fit in the text. From a teaching viewpoint, one possible complaint is that Anderson defines terms when needed, but he does not collect them in a glossary; this may cause frustration for some students.

Anderson takes care to give a self-contained presentation. A particularly notable example of this is his inclusion of proofs for two fundamental results in transversal theory. Given a collection of n sets, a *system of distinct representatives* (SDR, also called *transversal*) is a selection of n distinct elements, one from each set. The result popularly known as Hall's Marriage Theorem asserts that the obvious necessary condition is also sufficient: the union of any k of the sets must have at least k elements. This result is used in constructing chains to prove the equivalence of the LYM and normalized matching properties. An easy extension of it is in fact equivalent to Dilworth's Theorem; Anderson gives us one direction of the equivalence in addition to an independent proof of each.

The other result is less well known; it is a necessary and sufficient condition for the existence of a common SDR for two collections of sets. Griggs used this to show that if an LYM order has rank size $\{N_k\}$ that form a symmetric and unimodal sequence, then it has a symmetric chain decomposition. Anderson neglects to mention that this result gives the only known proof that the lattice of subspaces of a finite vector space, ordered by inclusion, is a symmetric chain order. Explicit construction of symmetric chain decompositions for these posets is still sought. Anderson misses another chance to tie the topics of the book together by not pointing out that a symmetric chain decomposition is a *completely saturated partition*, i.e., k-saturated for all k. The result just mentioned then suggests one of the most tantalizing open questions in this area; does every LYM order have a completely saturated partition?

Anderson says very little about lattices; in fact, he defines lattice and distributive lattice as an aside immediately *after* stating a result about them. Lattices can be avoided, if desired, but this makes it difficult to give an appreciation of the original proof of the Greene-Kleitman Theorem or why order ideals and the special posets in Sperner theory are of such interest. Some results, such as the Ahlswede-Daykin Inequality, reach their full generality and naturalness in the setting of distributive lattices. I have found it pedagogically helpful to include a unit on distributive lattices before presenting this inequality to students.

It is also possible that, in a long semester, an instructor may not want to work through all the details of extremal set theory, and instead include some related topics with a slightly different flavor. In addition to including some lattice theory, one can move on from the linear extensions of the XYZ Conjecture to discuss dimension theory of posets. Later, after becoming thoroughly familiar with subsets and binary vectors via the Kruskal-Katona Theorem, one can finish the course with a unit on coding theory. I will take this approach in my next graduate course.

In packing a well-developed subject into 250 pages, one must make choices; these will never please everyone, and quibbling wastes time. On balance, one is hard put to find complaints about this well-written and thorough book. It is a valuable addition to the literature, it will make it easy for interested mathematicians to acquire a new specialty, and it brings another area of mathematics into the accessible graduate curriculum.

DOUGLAS B. WEST
UNIVERSITY OF ILLINOIS

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Stochastic geometry and its applications, by D. Stoyan, W. S. Kendall, and J. Mecke, with a Foreword by D. G. Kendall. John Wiley and Sons, Chichester, New York, Brisbane, Toronto and Singapore, 1987, 345 pp., $49.95. ISBN 0471 90519 4

As a subject Stochastic Geometry surely existed (albeit anonymously) before this term first appeared in a title of a collection of papers [**1**] edited by E. F. Harding and D. G. Kendall in 1974. Numerous problems (and of course less numerous solutions), which in retrospect should be attributed to this field, have been discussed in countless papers scattered within journals and books too often devoted to nonmathematical applications and therefore obscure from the standpoint of a pure mathematician.

In many cases the authors of these papers were equipped merely with the tools of classical geometrical probability theory among which the notions of uniform distribution and independence were the basic. And yet their objectives were substantially more complicated concepts of what later came to be known as Stochastic Geometry. In the lucky cases the deficiency in tools was compensated by intuition.

Terminological ambiguity was quite widespread. For instance, within a paper considering random finite arrays of points the term "distribution" could simultaneously mean (a) the realization at hand, (b) the distribution of the typical point in the array, (c) a statistical estimate of (b), or (d) the distribution of the underlying point process. This terminological and conceptual mess