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Random mappings, by Valentin F. Kolchin. Optimization Software, Inc., distributed by Springer-Verlag, New York, 1986, xiv+206 pp., $ 80.00. ISBN 0-387-96154-2

In the past few decades we have seen an upsurge of articles and monographs dealing with the applications of probabilistic methods in various fields of mathematics. The best known of these fields are number theory, graph theory, and combinatorics. Probabilistic methods in combinatorics are the subjects of several books, such as P. Erdös and J. Spencer [**2**], V. F. Kolchin, B. A. Sevastyanov, and V. P. Chistyakov [**11**], and V. N. Sachkov [**18**]. V. F. Kolchin's new book is a welcome addition to this list.

The main topics covered in Kolchin's book are allocation (occupancy) problems, random permutations, random mappings, branching processes, random trees, and random forests. Allocation problems have already been discussed in great detail in the book by V. F. Kolchin, B. A. Sevastyanov, and V. P. Chistyakov [**19**], but most of the other topics are published here for the first time in monograph form.

Understanding the book requires only a basic knowledge of the elements of combinatorics and probability theory. The main analytic tool used in the book is the local limit theorem of B. V. Gnedenko [**4**], and some of its extensions. According to Gnedenko's theorem if $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ is a sequence of independent and identically distributed discrete random variables taking on integers only and if $E\{\xi_n\} = a$ ($|a| < \infty$), $\operatorname{Var}\{\xi_n\} = \sigma^2$ ($0 < \sigma < \infty$), and the greatest common divisor of the possible values of ξ_n is 1, then

$$\lim_{n\to\infty}\left[\sigma\sqrt{n}P\{\xi_1+\cdots+\xi_n=j\}-\phi\left(\frac{j-na}{\sigma\sqrt{n}}\right)\right]=0 \tag{1}$$

uniformly in j for $|j-na| < K\sqrt{n}$, $0 < K < \infty$. Here $\phi(x)$ is the normal density function, i.e.,

$$\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}. \tag{2}$$

In what follows we shall give a brief description of each topic covered in the book, mention some highlights of the results obtained, and add some historical remarks.

ALLOCATION (OCCUPANCY) PROBLEMS. Allocation problems have their origin in statistical physics (Maxwell-Boltzmann statistics) and date back to the nineteenth century. In the 1930s nonparametric statistical tests led to an interest in allocation problems.

The following model is investigated in the book: n balls are distributed in m boxes in such a way that all the m^n arrangements are equally probable. Denote by $\mu_r(m,n)$ ($r = 0, 1, \ldots, n$) the number of boxes containing exactly r balls.

One of the main results is the following: If $n = [m\alpha]$ where $0 < \alpha < \infty$, then for any $r = 0, 1, \ldots$ we have

$$\lim_{m\to\infty}\left[\sigma_r\sqrt{m}P\{\mu_r(m,n) = j\} - \phi\left(\frac{j - mp_r}{\sigma_r\sqrt{m}}\right)\right] = 0 \tag{3}$$

uniformly in j for $|j - mp_r| < K\sqrt{m}$, $0 < K < \infty$, where $\phi(x)$ is the normal density function, $p_r = e^{-\alpha}\alpha^r/r!$ and

$$\sigma_r^2 = p_r\left[1 - p_r - \frac{p_r}{\alpha}(r-\alpha)^2\right]. \tag{4}$$

In 1964 B. A. Sevastyanov and V. P. Chistyakov [19] proved that if $n = [m\alpha]$, $0 < \alpha < \infty$, and $m \to \infty$, then for every $s = 0, 1, \ldots$ the random variables $\mu_r(m,n)$ $(0 \le r \le s)$ have an asymptotic $(s+1)$-dimensional normal distribution. The above result (3) was found by V. F. Kolchin [8] in 1968 and he gave an elegant proof based on the local limit theorem of Gnedenko.

RANDOM PERMUTATIONS. Among the $n!$ permutations of $(1, 2, \ldots, n)$ a permutation is chosen at random in such a way that all the $n!$ permutations are equally probable. Denote by $\alpha(n)$ the number of cycles in this permutation. Then we have

$$P\{\alpha(n) = j\} = S(n,j)/n! \qquad (j = 1, 2, \ldots, n), \tag{5}$$

where $S(n,j)$ $(1 \le j \le n)$ is a Stirling number of the first kind. In 1942 V. Goncharov [6] proved that

$$\lim_{n\to\infty} P\left\{\frac{\alpha(n) - \log n}{\sqrt{\log n}} \le x\right\} = \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2/2}\,du. \tag{6}$$

In 1971 V. F. Kolchin [9] succeeded in proving the corresponding local limit theorem, that is, he proved that

$$\lim_{n\to\infty}\left[\sqrt{\log n}\;P\{\alpha(n) = j\} - \phi\left(\frac{j - \log n}{\sqrt{\log n}}\right)\right] = 0 \tag{7}$$

uniformly with respect to j in the interval $|j - \log n| < K(\log n)^{7/12}$. This result provides an asymptotic formula for $S(n,j)$, namely,

$$S(n,j) \sim \frac{n!}{\sqrt{\log n}}\phi\left(\frac{j - \log n}{\sqrt{\log n}}\right) \tag{8}$$

as $n \to \infty$ if $|j - \log n| < K(\log n)^{7/12}$.

In 1964 S. W. Golomb [5] observed that if L_n is the expected length of the longest cycle in a random permutation of $(1, 2, \ldots, n)$, then the limit $\lim_{n\to\infty}(L_n/n) = \lambda$ exists and his computation showed that $\lambda = 0.62432965\ldots$. In 1966 L. A. Shepp and S. P. Lloyd [20] proved that

$$\lambda = \int_0^\infty \exp\left\{-x - \int_x^\infty \frac{e^{-y}}{y}\,dy\right\} dx. \tag{9}$$

If we consider a permutation of $(1, 2, \ldots, n)$ as an element of the symmetry group S_n, then we can define the order of a permutation $g \in S_n$ as the smallest positive integer m for which $g^m = e$, the identity permutation. Let us choose a permutation at random among the $n!$ permutations of S_n so that all the

$n!$ permutations are equally probable. If ν_n denotes the order of the random permutation, then

$$\lim_{n\to\infty} P\left\{\frac{\log \nu_n - (\log^2 n)/2}{\sqrt{(\log^3 n)/3}} \le x\right\} = \Phi(x). \tag{10}$$

This result was obtained by P. Erdös and P. Turán [3] in 1967.

BRANCHING PROCESSES. Let us suppose that in a population initially we have i individuals and in each generation each individual reproduces, independently of the others, and has probability p_j $(j = 0, 1, \dots)$ of giving rise to j descendants in the following generation. Denote by η_n the number of individuals in the nth generation. The sequence $\{\eta_n; n \ge 0\}$ describes a so-called branching process. Define $\tau = \inf\{n\colon \eta_n = 0,\ n \ge 0\}$, i.e., τ is the time of extinction ($\tau = \infty$ if $\eta_n > 0$ for all $n \ge 0$), and

$$\rho = \eta_0 + \eta_1 + \cdots + \eta_n + \cdots, \tag{11}$$

i.e., ρ is the total number of individuals (total progeny) in the process (possibly $\rho = \infty$).

Several proofs in the book are based on the following basic theorem: If $\nu_1, \nu_2, \dots, \nu_n$ are independent random variables each having the same distribution $P\{\nu_1 = j\} = p_j$ $(j = 0, 1, \dots)$, then

$$P\{\rho = n | \eta_0 = i\} = (i/n)P\{\nu_1 + \cdots + \nu_n = n - i\} \tag{12}$$

for $1 \le i \le n$. In the book this theorem is proved in the context of branching processes (p. 104); however, it should be noted that (12) was proved originally in the context of queuing theory. (Cf. L. Takács [**22** and **23** p. 102].) The two versions are equivalent because, as D. G. Kendall [**7**] noted in 1951, the number of customers served in a busy period in a single server queue can also be interpreted as the total number of individuals in a branching process.

In making use of (12) V. F. Kolchin [**10**] proved that if $\{\eta_n; n \ge 0\}$ is a branching process for which $P\{\eta_0 = 1\} = 1$, g.c.d. $\{j\colon p_j > 0\} = 1$, $\sum_{j=0}^{\infty} jp_j = 1$, and $\sigma^2 = \sum_{j=0}^{\infty}(j-1)^2 p_j < \infty$, then

$$\lim_{n\to\infty} P\{\sigma\tau \le x\sqrt{n} | \rho = n\} = \sum_{k=-\infty}^{\infty} (1 - k^2x^2)e^{-k^2x^2/2} \tag{13}$$

for $x > 0$. The same limit distribution appears also in connection with random walks and order statistics. (Cf. L. Takács [**21**].)

RANDOM TREES AND FORESTS. A simple graph is called a forest if it has no cycles. A simple connected graph is called a tree if it has no cycles. Thus the components of a forest are trees. A rooted tree has one vertex, its root, distinguished from the other vertices. The height of a rooted tree is defined as the length of the longest path joining the root and another vertex. (The length of a path is the number of edges it contains.)

The number of distinct (labeled) trees with n vertices is n^{n-2}. This formula was found in 1899 by A. Cayley [**1**]. In 1918 H. Prüfer [**16**] gave a simple proof for Cayley's formula by observing that there is a one-to-one correspondence between the set of labeled trees having n vertices and the set of arrangements of $n-2$ balls in n boxes. This observation makes it possible to consider several problems for random trees as problems for random allocation. In addition, several problems concerning random trees can also be considered as problems in branching processes. (See R. Otter [**9**].)

There are n^{n-1} rooted trees with n labeled vertices. Let us choose a tree at random assuming that all the n^{n-1} trees are equally probable. Denote by τ_n the height of a random tree. In 1967 A. Rényi and G. Szekeres [**17**] proved that

$$\lim_{n\to\infty} P\{\tau_n \le x\sqrt{n}\} = \sum_{k=-\infty}^{\infty} (1 - k^2x^2)e^{-k^2x^2/2} \tag{14}$$

for $x > 0$. The result (13) is an extension of (14).

The author considers also forests having n vertices and consisting of k trees $(1 \le k \le n)$. There are kn^{n-k-1} such forests. Let us suppose that we choose a forest at random and all the possible kn^{n-k-1} forests are equally probable. In the book there are several theorems for the asymptotic distributions of the maximal tree size and the maximal tree height if $n \to \infty$ and $k \to \infty$. Most of these results were obtained by Yu. L. Pavlov [**13** and **14**].

RANDOM MAPPINGS. Let us suppose that n balls, numbered $1, 2, \ldots, n$, are distributed in n boxes, labeled $1, 2, \ldots, n$, and all the possible n^n arrangements are equally probable. We associate a random directed graph with the allocation process. Let $(1, 2, \ldots, n)$ be the vertex set of the graph. Two vertices i and j are joined by a directed edge (i, j) if ball numbered i is in box labeled j. If $j = i$, then there is a loop at vertex i. Each component of the graph contains either exactly one cycle or a loop. If we remove the edges in each cycle and remove all the loops, we get a forest of distinct trees whose roots are the vertices in the cycles and at the loops.

The author studies the asymptotic distributions of the number of trees, the sizes of the trees, and the height of the forest (maximal height of the trees) as $n \to \infty$. If τ_n denotes the height of the forest, then by a result of G. V. Proskurin [**15**] we have

$$\lim_{n\to\infty} P\{\tau_n \le x\sqrt{n}\} = \sum_{k=-\infty}^{\infty} (-1)^k e^{-k^2x^2/2} \tag{15}$$

for $x > 0$.

Kolchin's book is a comprehensive study of apparently diverse topics in combinatorics. In reality the various topics are strongly connected, allocation theory being the connecting element. The author's main aim is to unify the methods and results of the various topics. He has achieved his goal superbly.

REFERENCES

1. A. Cayley, *A theorem on trees*, Quart. J. Pure Appl. Math. **23** (1889), 376–378. [The Collected Mathematical Papers of Arthur Cayley, Vol. XIII, Cambridge Univ. Press, 1897, pp. 26–28.]

2. P. Erdős and J. Spencer, *Probabilistic methods in combinatorics*, Akadémiai Kiadó, Budapest, 1974.

3. P. Erdős and P. Turán, *On some problems of a statistical group theory*. III, Acta Math. Acad. Sci. Hungar. **18** (1967), 309–320.

4. B. V. Gnedenko, *On a local limit theorem in probability theory*, Uspekhi Mat. Nauk **3**, no. 3 (1948), 187–194. (Russian)

5. S. W. Golomb, *Random permutations*, Bull. Amer. Math. Soc. **70** (1964), 747.

6. W. Gontcharoff, *Sur la distribution des cycles dans les permutations*, C. R. (Doklady) Acad. Sci. URSS **35** (1942), 267–269.

7. D. G. Kendall, *Some problems in the theory of queues*, J. Roy. Stat. Soc. Ser. B **13** (1951), 151–185.

8. V. F. Kolchin, *A certain class of limit theorems for conditional distributions*, Litovsk. Mat. Sb. **8** (1968), 53–63. [English translation: Selected Translations in Mathematical Statistics and Probability **11** (1973), 185–197.] (Russian)

9. ——, *A problem of the allocation of particles in cells and cycles of random permutations*, Theor. Probability Appl. **16** (1971), 74–90.

10. ——, *Moment of degeneration of branching processes and height of a random tree*, Math. Notes **24** (1978), 954–961.

11. V. F. Kolchin, B. A. Sevastyanov, and V. P. Chistyakov, *Random allocations*, V. H. Winston and Sons, Washington, D.C., 1978.

12. R. Otter, *The multiplicative process*, Ann. Math. Statist. **20** (1949), 206–224.

13. Yu. L. Pavlov, *The asymptotic distribution of maximum tree size in a random forest*, Theor. Probability Appl. **22** (1977), 509–520.

14. ——, *Limit distributions of the height of a random forest*, Theor. Probability Appl. **28** (1984), 471–480.

15. G. V. Proskurin, *On the distribution of the number of vertices in strata of a random mapping*, Theor. Probability Appl. **18** (1973), 803–808.

16. H. Prüfer, *Neuer Beweis eines Satzes über Permutationen*, Arch. Math. u. Phys. **27** (1918), 142–144.

17. A. Rényi and G. Szekeres, *On the height of trees*, J. Austral. Math. Soc. **7** (1967), 497–507.

18. V. N. Sachkov, *Probability methods in combinatorial analysis*, "Nauka", Moscow, 1978. (Russian)

19. B. A. Sevastyanov and V. P. Chistyakov, *Asymptotic normality in the classical ball problem*, Theor. Probability Appl. **9** (1964), 198–211.

20. L. A. Shepp and P. S. Lloyd, *Ordered cycle length in a random permutation*, Trans. Amer. Math. Soc. **121** (1966), 340–357.

21. L. Takács, *Remarks on random walk problems*, Publ. Math. Inst. Hungar. Acad. Sci. **2** (1957), 175–182.

22. ——, *A generalization of the ballot problem and its application in the theory of queues*, J. Amer. Statist. Assoc. **57** (1962), 327–337.

23. ——, *Combinatorial methods in the theory of stochastic processes*, John Wiley and Sons, New York, 1967.

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Combinatorics of finite sets by Ian Anderson, Clarendon Press, Oxford, 1987, xv + 250 pp., $45.00. ISBN 0-19-853367-5

Combinatorics has come of age. Just as most children pay attention primarily to their own interests, so the mathematical endeavor in the early years produces seemingly unrelated results and problem solutions. Puberty can be viewed as the beginning of awareness about the rest of the world, and in mathematics this brings survey articles that pull results together and place them in a common context. Finally, with maturity comes patience, yielding