[Li] Chang-Shou Lin, *The local isometric embedding problem in* $\mathbf{R}^3$ *of two dimensional Riemannian manifolds with Gaussian curvature changing sign nicely*, Thesis, NYU, New York 1983.

[Mo] J. Moser, *A rapidly converging iteration method and non-linear differential equations*, Ann. Scuola Norm. Sup. Pisa (3) **20** (1966), 499–535.

[Na] J. Nash, *The embedding problem for Riemannian manifolds*, Ann. of Math. (2) **63** (1956), 20–64.

[Ni] L. Nirenberg, *On a problem of Hans Lewy*, Uspekhi Mat. Nauk **292** (176) (1974), 241–251.

[Sc] J. Schwartz, *On Nash's implicit functional theorem*, Comm. Pure Appl. Math. **13** (1960), 509–530.

[Se] F. Sergeraert, *Une generalisation du théorème des fonctiones implicites de Nash*, C. R. Acad. Sci. Paris Ser. A **270** (1970), 861–863.

[Sp] M. Spivak, *Differential geometry*, Volume 5, Publish or Perish, Boston, 1975.

[We] J. Weingarten, *Über die Theorie der aufeinander abwickelbaren Oberflachen*, Berlin, 1884.

[Wy] H. Weyl, *Cartan on groups and differential geometry*, Bull. Amer. Math. Soc. **44** (1938), 598–601.

[Ya] D. Yang, *Involutive hyperbolic differential systems*, thesis, Harvard Univ., Cambridge, Mass., 1982.

HOWARD JACOBOWITZ
RUTGERS UNIVERSITY

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 19, Number 2, October 1988

0273-0979/88 $1.00 + $.25 per page

Representations of rings over skew fields, by A. H. Schofield. London Mathematical Society Lecture Note Series, vol. 92, Cambridge University Press, Cambridge, London, New York, New Rochelle, Melbourne, Sydney, 1985, xii+223 pp., $27.95. ISBN 0-521-27853-8

Let k be a commutative field. For an associative k-algebra R with unit, a *representation* of R is a k-algebra homomorphism from R to a matrix ring $M_n(S)$ where S is also a k-algebra. To be precise, this is an n-dimensional representation of R over S. In the historical beginnings of this notion, S was assumed to be k, but it was soon seen to be useful to allow S to be an extension of k, chosen perhaps to decompose the representation. When R is finite k-dimensional and semisimple (the word semiprimitive is also used) the Wedderburn classification theorem states that R is a direct product of simple Artinian rings, i.e., those of the form $M_n(D)$ for D a skew field (division algebra) over k. Thus it is worthwhile to allow S to be a skew field in the definition of representation. While in the above case D will be finite k-dimensional, for infinite-dimensional R we may need to allow the most general notion of representation over any skew field.

Classically, representations $R \to M_n(D)$ have been considered equivalent if they are conjugate, i.e., if there is a conjugation (inner) isomorphism of $M_n(D)$ which carries one representation to the other. This is not sufficient in the new generality: not only might n and D be different for the two representations, but also the algebra $M_n(D)$ may have other isomorphisms. This is even true for $n = 1$, so we examine that case.

Let $R \to D$ be a k-homomorphism to a skew field D. Nonzero elements of the image $R_0 \subseteq D$ will have inverses in D and we may form the subalgebra $R_1 \subseteq D$ generated by R_0 and all these inverses. Continuing, we can form the subalgebra $R_i \subseteq D$ generated by R_{i-1} and the inverses in D of nonzero elements of R_{i-1}. The union of the R_i forms a skew subfield R_D of D, generated as a skew field by R_0. In this case $h\colon R \to R_D$ is an epimorphism of k-algebras, i.e., for any k-algebra T and k-homomorphisms $i, j\colon R_D \to T$ such that $ih = jh$, we have $i = j$. It was the suggestion of P. M. Cohn to say that h determines the representation, in the sense that two representations $R \to D$ and $R \to E$ are equivalent if $R_D \cong R_E$ in such a way as to make $R \to R_D \cong R_E$ commute. Or equivalently that one-dimensional skew field representations $R \to D$ and $R \to E$ are equivalent if there is a skew field F and a commutative diagram

$$\begin{tikzcd} & & D \\ R \arrow[r] \arrow[urr] \arrow[drr] & F \arrow[ur] \arrow[dr] & \\ & & E. \end{tikzcd}$$

When R is a commutative the skew field R_D generated by the image of R in D is already R_1, a commutative field that consists of fractions of the form $ab^{-1} = b^{-1}a$. To each one-dimensional representation $R \to D$ there corresponds one prime ideal of R, the kernel. The reverse of this correspondence takes a prime ideal P of R to the representation of R in the field of fractions of R/P. Thus the set $\operatorname{Spec} R$ of prime ideals of R determines the one-dimensional representations of R up to this notion of equivalence.

For R Noetherian a similar situation occurs. To each one-dimensional skew field representation $R \to D$ there corresponds the kernel P, a completely prime ideal (i.e., $ab \in P$ implies $a \in P$ or $b \in P$). The reverse correspondence is accomplished by forming R/P, a Noetherian domain, which by Goldie's theorem has a skew field of quotients whose elements are of the form ab^{-1}. We could again call the set of completely prime ideals $\operatorname{Spec} R$, though this is not standard.

For general R, there are significant differences. First the infinite chain $R_0 \subseteq R_1 \subseteq \cdots \subseteq R_D$ may actually be proper, as in the case of $(xy^{-1}+yx^{-1})^{-1}$ where x and y have no common right or left multiple. Second, the kernel of the homomorphism may not determine the representation. As an example, consider the free k-algebra $k\langle X\rangle$ on a set X of indeterminates (the noncommutative polynomial ring with central coefficients from k). This has been embedded in various ways (with the same trivial kernel) into algebras that have classical skew fields of fractions, but all have certain rational equations in the indeterminates (involving inverses). P. M. Cohn set out to form a "free skew field" $k\langle\!\langle X\rangle\!\rangle$, a skew field generated by $k\langle X\rangle$ and having the fewest rational equations among the indeterminates. He needed to characterize one-dimensional representations in terms of information on R as in the previous cases.

During the 60s his efforts met with nearly complete success. The set of equivalence classes of 1-dimensional skew field representations of R is essentially the set of k-homomorphisms $R \to D$ where D is generated as skew field by the image of R. G. M. Bergman has suggested the same Sfield-Spec R for this set (also made into a category and topological space). To each $R \to D$

of Sfield-Spec R there corresponds the set P of square matrices (of all sizes) that become singular as matrices over D. Cohn called such P a "prime matrix ideal" of R. The reverse correspondence required an abstract definition of a prime matrix ideal P over R and a construction of an element of Sfield-Spec R from P. Every element of the constructed skew field has the form of an entry in the inverse of a square matrix over R that is *not* in P.

Various other characterizations of Sfield-Spec R have been given; we describe one. A "Sylvester matrix rank function" on R (terminology of A. H. Schofield) is a function ρ defined on matrices over R and having real values, such that the following axioms hold:

1. $\rho(I) = 1$, where I is the 1-by-1 identity matrix;
2. $\rho\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = \rho(A) + \rho(B)$;
3. $\rho\left(\begin{smallmatrix} A & 0 \\ C & B \end{smallmatrix}\right) \geq \rho(A) + \rho(B)$;
4. $\rho(AB) \leq \min(\rho(A), \rho(B))$

for all matrices A, B, C of appropriate sizes over R. For each element $R \to D$ of Sfield-Spec R we can obtain such a rank function ρ by letting $\rho(A)$ be the rank of the image of the matrix A over D. Then this gives a 1-1 correspondence between Sfield-Spec R and the set of Sylvester matrix rank functions on R with values in the nonnegative integers. For the free algebra $k\langle X\rangle$ the free field $k \langle\!\langle X \rangle\!\rangle$ corresponds to the "inner" rank function ρ defined as follows: for an $m \times n$ matrix A, $\rho(A)$ is the minimum k such that A factors as the product of a $m \times k$ and a $k \times n$ matrix over R.

Now for representations of dimension $n > 1$ there are significant differences. For a representation $R \to M_n(D)$ there is no analog for the process of adjoining inverses of elements of the image $R_0 \subseteq M_n(D)$ to form R_1, as before. Thus we do not get an appropriate (or unique) subobject generated by R_0 and hence no analogous way to identify equivalent representations. Even when R is right Noetherian and for each prime ideal P of R we can use Goldie's theorem to construct a simple Artinian right ring of quotients of R/P, this will only classify the so-called "left-flat" epimorphisms $R \to M_n(D)$. For an arbitrary representation the kernel need not be prime nor enough to determine the representation.

Bergman has suggested some possibilities for resolving these difficulties. He noted that given a representation $R \to M_n(D)$ we can get a Sylvester matrix rank function on R by defining the rank of a $p \times q$ matrix A over R to be $\frac{1}{n}$ times the rank of the $np \times nq$ image matrix of A over D. The values of this rank function are in $\frac{1}{n}Z$, but there is no immediate way to get from such a rank function to a representation. Bergman also suggested a new definition of equivalence, referring to the $M_n(D)$ as simple Artinian k-algebras, without specifying n. His proposal was that we regard two representations $R \to S_1$, $R \to S_2$ with S_i simple Artinian as equivalent if there is a simple Artinian S_3 and maps making a commutative diagram

$$\begin{tikzcd} & S_1 \arrow[dr] & \\ R \arrow[ur] \arrow[dr] & & S_3 \\ & S_2 \arrow[ur] & \end{tikzcd}$$

This sort of "pushout" diagram is quite different from the "pullback" diagram we earlier gave as a definition of equivalence for 1-dimensional representations. Are we to give up that earlier notion?

Let's digress briefly to see why these definitions are equivalent. Since homomorphisms (preserving 1) of skew fields are embeddings, suppose we have a pullback skew field F regarded as a skew subfield of two skew fields D and E. We need to embed both D and E into a larger (pushout) skew field. This may be done using the coproduct of D and E over F (also called the free product amalgamating F), which is obtained by giving generators and relations for D and E over F and then forming the ring generated over F by the union of the generators of D and E, subject (only) to the union of the relations. The result is an algebra containing copies of D and E but having no relations between them other than those of F. Cohn had considered this earlier and shown that this algebra is a "semifir," one of the types of algebras for which the inner rank defined previously is a Sylvester matrix rank function. From this we get a faithful representation (embedding) of the coproduct into a skew field, as needed. Cohn called this large skew field the "skew field coproduct" of D and E over F.

Even though we can reconcile Bergman's proposed definition of equivalence with our earlier 1-dimensional definition, there still are large obstacles. In particular, it is not even clear why the proposed relation satisfies transitivity. Also, there is no evident reason why equal rank functions should yield equivalent representations.

Nevertheless, in the book under review Schofield carries out most of Bergman's program. As suggested, the main result is a 1-1 correspondence between the equivalence classes of n-dimensional representations and the set of Sylvester matrix rank functions on R with values in $\frac{1}{n}Z$. Many general constructions and other generalizations are described along the way, and several open questions are answered.

The major tools are extension of rank functions and universal localization. Given rank functions on two appropriate algebras, Schofield uses Bergman's coproduct structure theorems to extend the rank functions to a rank function on the coproduct. Then he carries out a universal localization of the coproduct, adjoining coefficients for universal inverses of certain maps of projective modules. Finally the rank function may be extended to the localized ring. Given two simple Artinian algebras S_1, S_2 with common simple Artinian subalgebra S, the above procedure results in a simple Artinian algebra containing both S_1 and S_2, the "simple Artinian coproduct" of S_1 and S_2 over S. This coproduct is a ring of matrices of size equal to the least common multiple of the sizes of the matrices for S_1 and S_2. This shows transitivity of the equivalence relation.

Again using extension and localization, the author demonstrates the 1-1 correspondence between representations and rank functions. Then he develops the notion of an "indecomposable" representation (though not in those words) and obtains each representation in a unique way from indecomposable ones. Along the way he shows (for example) that every rank function with real values on a hereditary ring is induced from a homomorphism to a von Neumann regular ring, and that every Artinian ring embeds in a simple Artinian ring.

A large set of tools having been developed for the above work, the author proceeds to apply much of it to questions about commutative and skew field subalgebras of the skew field coproduct and the simple Artinian coproduct. In particular, he shows that even if two skew fields contain no elements algebraic over the center, the skew field coproduct of the two over the center may contain a (commutative) algebraic subfield. However, the transcendence degree of subfields of a skew field coproduct cannot be much bigger than those of the factors. He also shows that free skew fields on different numbers of generators cannot be isomorphic.

Finally, by introducing a generalization of the skew field coproduct, Schofield gives a complete solution to Artin's problem for skew fields: for any integers $m, n > 1$ he constructs skew fields $D \subseteq E$ such that E has right D-dimension m and left D-dimension n. This allows him to construct a hereditary Artinian ring of (finite) representation type $I_2(5)$ as suggested by Dowbor, Ringel, and Simson.

The book is definitely a work of power and of breadth, with much to interest a wide class of ring theorists. The reader is given the strong impression that mathematics is in the process of being done here. Perhaps inevitably in such a case, there are many misprints, some of which are serious. The most important are described in Bergman's review (MR 87c:16001). The reader should also be warned that the index is of limited usefulness, since it has only a selection of the important subjects.

REFERENCES

G. M. Bergman, *Modules over coproducts of rings*, Trans. Amer. Math. Soc. **200** (1974), 1–32.

P. M. Cohn, *Free rings and their relations*, London Math. Soc. Monographs 19, Academic Press, New York, 1985.

——, *The embedding of firs in skew fields*, Proc. London Math. Soc. **23** (1971), 193–213.

P. Dowbor, C. M. Ringel, and D. Simon, *Hereditary artinian rings of finite representation type*, Representation Theory II, Lecture Notes in Math., vol. 832, Springer-Verlag, Berlin and New York, 1980.

P. Malcolmson, *Construction of universal matrix localizations*, Advances in Noncommutative Ring Theory, Lecture Notes in Math., vol. 951, Springer-Verlag, Berlin and New York, 1982.

——, *Determining homomorphisms to skew fields*, J. Algebra **64** (1980), 399–414.

B. Stenstrom, *Rings of quotients*, Grundlehren 217, Springer-Verlag, Berlin and New York, 1975.

PETER MALCOLMSON
WAYNE STATE UNIVERSITY

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 19, Number 2, October 1988

0273-0979/88 $1.00 + $.25 per page

Partial differential equations: An introduction to a general theory of linear boundary value problems, by Aleksei A. Dezin. Translated by Ralph P. Boas. Springer-Verlag, Berlin, Heidelberg, New York, xii+163 pp., $82.00. ISBN 3-540-16699-8