on the subject and at some of the additional references listed below or in the excellent bibliography at the end of Lehto's book.

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Kleinian groups and uniformization in examples and problems, by S. L. Krushkal′, B. N. Apanasov, and N. A. Guseviskiĭ. Translated from Russian by H. H. McFaden. Translations of Mathematical Monographs, vol. 62, American Mathematical Society, Providence, R.I., 1986, vii+198 pp., $66.00. ISBN 0-8218-4516-0

The old order changes; classical divisions of mathematics into subject areas of distinguishable type have been progressively refined and fragmented until the attempt to classify a research paper via the MR subject index appears as a task of comparable size to understanding the results themselves. This Balkanisation process is compounded by an increasing—and no doubt welcome—tendency towards federalisation of the ideas and techniques which erodes and transcends even the ancient divides of algebra, analysis, and geometry.

How, for instance, should one approach Kleinian groups? As discrete subgroups of the Lie group of complex two-by-two matrices, Kleinian groups fit naturally within at least four broad subject areas, reflecting their origins within the classical analysis, the underlying (abstract) group-theoretical structures which they represent, their position within the deformation theory of discrete groups in general, and the topological connection with hyperbolic three-dimensional manifolds first noticed by Poincaré and recently brought back to prominence by Thurston's revolutionary ideas. None of which mentions the specific and important links with number theory, algebraic groups, the geometry of algebraic curves and their moduli spaces, or the analogy with rational maps on the sphere and conformal dynamics. Pity, then, the would-be expositor of a subject with such broad-based appeal, who aims for a complete introductory account. The recent substantial text by Beardon [**Be**], which concentrates on the elementary geometric theory of Möbius transformations in the plane, emphasizes the scale of the problem.

We outline briefly the historical development and present position of Kleinian groups in order to appreciate their scope. From Poincaré's initial work [**204**], where the background is the analytic continuation of solutions to linear differential equations in the complex plane, through the ensuing development of uniformisation theory for Riemann surfaces, the emphasis lay in the action of $\mathrm{SL}_2(\mathbf{C})$ on the sphere (or projective line) by Möbius transformations

$$z \mapsto \frac{az+b}{cz+d}.$$

Poincaré realised that this affords a link with three-dimensional geometry by extending the action on the sphere to the open 3-ball H^3 lying inside the sphere, the extended transformations being isometries of H^3 in the non-Euclidean ("hyperbolic") metric on it. This action can also be obtained by the identification of H^3 with the principal homogeneous space of the Lie group $\mathrm{SL}_2(\mathbf{C})$. In this setting, it is possible to construct examples of compact 3-manifolds, arising as the quotient space of H^3 by a discrete group of isometries; for instance, one may consider the group generated by transformations which identify opposite faces of a certain regular dodecahedron (of appropriate size) inside the ball. This is a precursor of the modern theory of hyperbolic structures on 3-manifolds.

Until recently, the mainstream of development has been the analytic study of Riemann surfaces and the variety of ways to represent them as quotient spaces of some open subset Ω of the sphere by a Kleinian group G, operating so as to preserve Ω and act *discontinuously* (so that the G-orbit of any point in Ω is discrete). Such a representation is called a uniformisation of the surface. The existence of such representations—with the property that Ω is the unit disc—for any compact Riemann surface of genus at least two is the content of the uniformisation theorem of Klein, Poincaré, and Koebe; the latter gave a complete proof of this epoch-making result many years after the original discovery. Kleinian groups which preserve a Euclidean disc are usually called Fuchsian groups.

In the years after the Second World War, the deformation theory built up by Ahlfors, Bers, and their students from foundational ideas of Teichmüller came to be the primary tool in the study of Kleinian groups, affording a description in complex-analytic terms of their moduli spaces, which parametrise the variety of Kleinian groups whose structure and action on the sphere conforms to a specified topological pattern or marking. The key idea is the notion of quasiconformal homeomorphism, a precisely controlled weakening of conformality, which performs the task of deforming a Kleinian group by conjugating it (within the group of all homeomorphisms of the sphere); the resulting group will under suitable restrictions on the conjugating homeomorphism still be Kleinian, that is, consist of Möbius transformations. These quasiconformal mappings possess marvellous flexibility and permit analytic control of the local variational theory. They also admit wide application elsewhere in complex function theory, for instance in the work of Sullivan and of Douady and Hubbard on iteration of holomorphic maps on the sphere and in the theory of univalent functions.

Group-theoretical properties of Kleinian groups, and the way in which these are manifested in the topological structure of the group action on sets Ω where it acts discretely, have been studied in detail by Bernard Maskit, who developed geometrical versions of the standard constructions in combinatorial group theory, enormously extending the original construction by Felix Klein of free products of group actions. These methods generate a wealth of fascinating examples and are an essential step in building more complicated groups from standard types—"designer Kleinian groups" can be tailored and assembled to a prearranged pattern.

A dramatic change in emphasis occurred around 1975 with the introduction by Thurston of radically new ideas from topology of 3 manifolds and dynamical systems on surfaces. A deep study by Troels Jørgensen into the special class of Kleinian group that occurs as a quasiconformal deformation of a Fuchsian group uniformising a once-punctured torus had yielded (by a limiting procedure) a particularly interesting new example of Kleinian group [**110**]; this has the property that the corresponding quotient manifold produced from H^3 is compact and has the topological structure of a fibre space, with base the circle S^1 and fibre over each point a torus. One of Thurston's crucial insights is that any compact 3-manifold that has no reasonable excuse, in the sense that certain elementary topological properties of hyperbolic manifolds are not violated, should possess a hyperbolic quotient space structure. This includes the manifolds obtained by the so-called mapping-torus construction from a surface homeomorphism: if $f\colon S \to S$ is a self-homeomorphism of a surface with negative Euler characteristic, which mixes up the topology of S sufficiently (no finite collection of loops in S is preserved up to isotopy by f), then the manifold $S \times [0,1]/\sim$ with $\sim$ the identification of points $(x,0)$ with $(f(x),1)$ for $x \in S$, which is the mapping torus $M(f)$ of f, carries a hyperbolic structure; the examples of Jørgensen fit into this framework.

Thurston's work brings Kleinian groups into the limelight of the theatre of three-dimensional topology. For a detailed introductory account of these exciting developments, one may consult Thurston's growing list of papers beginning with the survey [**303**]; a good account of his geometrisation conjecture is given by Peter Scott [**Scott**], and an invaluable survey of Thurston's hyperbolic uniformisation theorem for Haken 3-manifolds may be found in John Morgan's chapter of the Smith Conjecture Symposium [**Bass**]. Unfortunately, the very influential lecture notes by Thurston [**234**] are still unpublished; they provide the essential new ideas and examples necessary to penetrate the Byzantine subtleties of Kleinian 3-manifold topology. Some portions have been carefully worked out in detail in the conference proceedings [**Epstein**]; others may be found as background in an important recent paper of F. Bonahon [**Bon**].

The geometry of manifolds in general has received a consequent stimulus, and a stream of results on hyperbolic, Lorenzian, affine, and projective structures including analogues of uniformisation has gratifyingly enriched the collection of interesting explicit examples, a valuable aid to understanding for the uninitiated. This brings us to the book in hand, which seeks to exploit that basic resource to an unusual extent. It presents theory by way of examples, with illustration of concepts replacing the traditional parade of theorem and proof, and with a consequent reduction of the exposition problems discussed earlier to negligible proportions. Some basic facts on Kleinian groups are sketched, with a reference to original articles for more detail, to be followed by a more serious introduction to the notion of Teichmüller space and some representative uniformisations of Riemann surfaces; a terse summary of higher-dimensional manifolds ends the theoretical part at sixty pages. The core of the text is the unusually large compendium of examples and subsequent discussion, which presents the reader with both illustration and elaboration of the theory. The extraordinary richness of structure involved in this geometric group theory is amply reflected in the range of geometric, algebraic, and topological properties to be found experimentally here; the reader must participate by carrying through the detailed verifications which are often necessary to supplement the commentary, but the effort involved will be amply repaid and there is usually a reference to the original source as a safety net.

The final third of the book consists of problems, intended as strengthening and extension of the general theory. It attempts to blend the standard type of problem with more demanding questions arising from the known theory, and the level of difficulty fluctuates alarmingly in places, with certain questions of unexpected (and unexplained) complexity mingled with others of a routine nature. No doubt a group of students working in co-operation under the benevolent eye of an expert would derive benefit from toiling through them, but a solitary reader unversed in the intricacies of 3-manifold topology could be better served—certainly one should heed the cautionary note sounded in the editor's preface and echoing through the footnotes. However, it must also be said that the authors have provided a laudably extensive bibliography and, by their selection of topics, have succeeded in making accessible a range of interesting research problems, some of which are listed in a final section.

Certainly one can learn much about Kleinian groups from this book and also, perhaps, something about the exposition of mathematics. On geometrical matters, learning by examples seems to work well; often the observer can find an independent view to the conventional one, and in any case the approach to structure through geometrical models is inherently rather experiential. The real shortcoming of the text is its failure to confront the topological difficulties, which are already apparent in two dimensions; but then one should not be too harsh here since no text currently available deals adequately with this aspect of discrete group theory. I award the authors a quiet cheer for their enthusiastic coverage of the menagerie of examples; but they have left us to write the guide book ourselves.

For the convenience of the reader, I end this review with a list of (mostly recent) supplementary references; in the review, references by number refer to the bibliography in the text.

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Finally, for a demonstration of how to weave a spell with geometry and analysis on hyperbolic space, I recommend the beautiful article by F. Apéry in Gazette des Mathematiciens (1982), pp. 57–86, entitled *La baderne d'Apollonius*, which also provides some illustrations of the wide opportunities for computer graphics in this field, beyond the current obsession with fractal curves.

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Teichmüller theory and quadratic differentials, by Frederick P. Gardiner. John Wiley and Sons, New York, Chichester, Brisbane, Toronto, Singapore, 1987, xvii+236 pp., $46.95. ISBN 0-471-84539-6

The theory of moduli of Riemann surfaces occupies a central role in modern mathematics. Its origins lie in the classical theory developed in the nineteenth century, and it has attracted the attention of many of the outstanding mathematicians of the twentieth century, including Poincaré and Hilbert at the beginning of the century, Ahlfors and Bers during most of the middle half of this century, and Thurston and Sullivan at the present time. The subject is rich with deep general theories and full of interesting special cases. It has a technology of its own, but borrows extensively from other disciplines (topology, algebraic geometry, several complex variables) and has applications to diverse fields (partial differential equations, minimal surfaces, particle physics).

One of the main objects in the theory is the Teichmüller space $T(p,n)$ whose points are all the "marked" compact Riemann surfaces of genus p with n punctures or distinguished points. To avoid the elementary and easy to handle cases, one assumes that the surface has negative Euler characteristic; that is, both p and n are nonnegative integers with $2p-2+n > 0$. By a marking on a surface, we mean a choice of a basis for the fundamental group of the surface. It is an important observation that the space of marked surfaces is easier to study than the more natural object $R(p,n)$ consisting of conformal