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Geometry and arithmetic around Euler partial differential equations, by R.-P. Holzapfel. D. Reidel Publishing Company, Dordrecht, Boston and Lancaster, 1986, 184 pp., \$34.50. ISBN 90-277-1827-X

The classical theory of the hypergeometric function, as created by a host of mathematical luminaries, is rich in its diverse applications to such mathematical specialties as representation theory, the theory of automorphic forms, elliptic curves and integrals, and number theory. This book is concerned with an equally rich study in two dimensions and focuses on specific examples. As the author indicates, it is often easier to develop a general theory than to understand concretely some mathematical object. Yet, in the final analysis, such completely understood concrete objects are the wellsprings of future mathematics.

The monograph is not self-contained, but where precise references are unavailable the author develops the necessary techniques to facilitate a natural understanding of the results and their proofs. The reader should certainly be versed in some of the basics of algebraic geometry as found in Shafarevich [**6**], Griffiths and Harris [**1**], or Hartshorne [**2**].

Many of the results are new and concern differential equations of the type

$$0 = \frac{\partial^2}{\partial t_1 \partial t_2} F + \frac{1}{t_2 - t_1}\left(a\frac{\partial}{\partial t_1} - b\frac{\partial}{\partial t_2}\right) F, \qquad a, b \text{ constant},$$

which were called Euler partial differential equations by Darboux. These are known from acoustics. Picard discovered some solutions coming from certain algebraic families of complex curves depending on two parameters. These solutions are expressed as integrals

$$\int \frac{dx}{\sqrt[n]{Q(x, t_1, t_2)}}$$

on the corresponding Riemann surfaces

$$Y^n = Q(X, t_1, t_2)$$

with $Q(X, t_1, t_2)$ polynomials in X depending on two complex parameters. Picard also noted a connection between the monodromy of these functions, (arithmetic) lattices acting on the two-dimensional complex unit ball, and automorphic functions on the ball.

In Chapter II, the author obtains results about general systems of partial differential equations of Euler-Picard type in arbitrary dimensions and solutions of the type

$$\int \frac{dx}{\sqrt[n]{Q(x)}}$$

where Q depends on the appropriate parameters. He connects these with automorphic forms with respect to Eisenstein lattices in the two-dimensional complex unit ball for the special case of "Picard curves"

$$Y^3 = p_4(X)$$

($p_4(X)$ a polynomial of degree four in X). The theme of arithmetic ball quotient surfaces and the classification of complex algebraic surfaces is at the heart of the first chapter.

As an example, consider the following special case of Picard's [5] hyperfuchsian functions

$$I(g,h;t_1,t_2) = \int_g^h \frac{dx}{\sqrt[3]{x(x-1)(x-t_1)(x-t_2)}}$$

$g, h \in \{0, 1, t_1, t_2, \infty\}$. These can be shown to satisfy an Euler partial differential equation. Three of them form a fundamental set of solutions to a system of three partial differential equations. They serve to map a dense subset of $\mathbf{P}^2$ (locally injectively) into the two-dimensional complex unit ball. The monodromy group is generated by five elements of $\mathbf{U}((2,1);O_K)$ where $K = \mathbf{Q}(\sqrt{-3})$; see also Mostow and Deligne [4]. These five elements generate the fundamental group of the space P of nonsingular points of the differential equations

$$P = \{(t_1,t_2) \in \mathbf{C}^2,\ t_i \neq 0, 1,\ t_1 \neq t_2\},$$

which is also the space of "distinguished" smooth Picard curves

$$O(t_1,t_2)\colon Y^3 = X(X-1)(X-t_1)(X-t_2).$$

The author proves that the monodromy group is the principal congruence subgroup $\widetilde{\Gamma}'$ for the ideal $(1-\omega)O_K$, $\omega = e^{2\pi i/3}$, of the full Eisenstein lattice of the ball $\widetilde{\Gamma} = \mathbf{U}((2,1),O_K)$. He investigates in detail the rings of automorphic forms of $\widetilde{\Gamma}'$ and $\widetilde{\Gamma}$ and the geometry/uniformization of $B/\widetilde{\Gamma}'$ and the Baily-Borel compactification $\widehat{B/\widetilde{\Gamma}}$.

The second chapter focuses on the Euler-Picard system of partial differential equations and the Gauss-Manin connection for the family

$$Y^n = X(X-1)(X-t_1)\cdots(X-t_r) \tag{1}$$

generalizing Picard's results in the two-parameter case. Here the author draws on Katz [3]. He investigates the structure of the Gauss-Manin connection not only in case (1) but more generally for the cycloelliptic curve families of the type

$$Y^n = (X-1)^{b_{-1}} X^{b_0} (X-t_1)^{b_1} - (X-t_r)^{b_r},$$

the b_i being natural numbers.

For readers intrigued by the interplay between algebraic geometry, automorphic forms, and the monodromy of differential equations, this book will be a delight. For those with a basic background desiring a further initiation into these topics, the concrete examples will provide a welcome focus.

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Lie groupoids and Lie algebroids in differential geometry, by Kirill Mackenzie. London Mathematical Society Lecture Note Series, vol. 124, Cambridge University Press, Cambridge, 1987, xv + 327 pp., $34.50. ISBN 0-521-34882-X

It is worthwhile to first examine what the author has to say about groupoids and about his book:

> The concept of groupoid is one of the means by which the twentieth century reclaims the original domain of application of the group concept. The modern, rigorous concept of group is far too restrictive for the range of geometrical applications envisaged in the work of Lie. There have thus arisen the concepts of Lie pseudogroup, of differentiable and of Lie groupoid, and of principal bundle—as well as various related infinitesimal concepts such as Lie equation, graded Lie algebra and Lie algebroid—by which mathematics seeks to acquire a precise and rigorous language in which to study the symmetry phenomena associated with geometrical transformations which are only locally defined.
>
> This book is both an exposition of the basic theory of differentiable and Lie groupoids and their Lie algebroids, with an emphasis on connection theory, and an account of the author's work, not previously published, on the abstract theory of transitive Lie algebroids, their cohomology theory, and the integrability problem and its relationship to connection theory. [p. vii]

> The primary aim of this book is to present certain new results in the theory of transitive Lie algebroids, and in their connection and cohomology theory; we intend that these results establish a significant theory of abstract Lie algebroids independent of groupoid theory. As a necessary preliminary, we give the first full account of the basic theory of differentiable groupoids and Lie algebroids, with emphasis on the case of Lie groupoids and transitive Lie algebroids. [p. ix]