BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 19, Number 1, July 1988

0273-0979/88 $1.00 + $.25 per page

Functional calculus of pseudo-differential boundary problems, by Gerd Grubb. Progress in Mathematics, vol. 65, Birkhäuser, Boston, Basel, Stuttgart, 1986, vi + 511 pp., $49.00. ISBN 0-8176-3349-9

It is well known that the parametrix of an elliptic differential operator on a manifold without boundary will be a pseudodifferential operator. When one considers an elliptic boundary value problem the parametrix certainly contains a pseudodifferential operator in the domain, but other terms arise because of the presence of the boundary. In the case of manifolds without boundary it is useful to consider an algebra of pseudodifferential operators that contain both the elliptic differential operators and their parametrices. The study of the index of an elliptic operator is one of the applications of such algebra.

Analogously one may try to construct an algebra of operators on a manifold with boundary that contains both elliptic boundary value problems and their parametrices. The first question to answer is the following: how to define a Fredholm boundary value problem for a pseudodifferential operator on a manifold with a boundary?

Consider first an elliptic boundary value problem in the domain $\Omega \subset \mathbf{R}^n$:

$$L(x,D)u = f, \qquad x \in \Omega, \tag{1}$$

$$p_{\partial\Omega} B(x,D)u = g, \tag{2}$$

where $p_{\partial\Omega}$ is the restriction operator to $\partial\Omega$, and $L(x,D), B(x,D)$ are, in general, matrices of differential operators. "Freeze" the coefficients of (1), (2) at an arbitrary point $x' \in \partial\Omega$, introduce a system of coordinates in a neighbourhood of x' such that the equation of $\partial\Omega$ will be $x_n = 0$, and, ignoring the dependency of coefficients on x', take the Fourier transform in variables tangential to $\partial\Omega$. Then one can associate with (1), (2) a family of boundary value problems for ordinary differential equations on the half-line

$$L_0(y, 0, D_n)v(x_n) = f_0(x_n), \qquad 0 < x_n < +\infty, \tag{1'}$$

$$B_0(y, x_n, D_n)v(x_n)|_{x_n=0} = g_0, \tag{2'}$$

where $y = (x', \xi') \in T^*(\partial\Omega)$, $|\xi'| = 1$, and $L_0(y, x_n, \xi_n)$, $B_0(y, x_n, \xi_n)$ are symbols of the principal parts of L, B written in coordinates (x', x_n). $T^*(\partial\Omega)$ is the cotangent bundle of $\partial\Omega$. The investigation of (1′), (2′) is the crucial step in the study of boundary value problem (1), (2). Note that (1′) defines a Fredholm operator from $H_s(\mathbf{R}^1_+)$ to $H_{s-2m}(\mathbf{R}^1_+)$, where $2m$ is the order of L_0 and $H_s(\mathbf{R}^n_+)$ is the Sobolev space in $\mathbf{R}^1_+$. This Fredholm operator has no cokernel and the dimension of its kernel is exactly the number of boundary conditions (2′). Indeed the role of the boundary conditions (2) is to "kill" the kernel of (1′).

Consider now a pseudodifferential equation in $\Omega \subset \mathbf{R}^n$:

$$p_\Omega A(x,D)u_0 = f, \tag{3}$$

where $A(x,D)$ is a pseudodifferential operator in $\mathbf{R}^n$, $\operatorname{supp} u_0 \subset \overline{\Omega}$ and p_Ω is the restriction operator to Ω. Analogously to (1′) one can associate with (3) a family of pseudodifferential operators on the half-line:

$$(3') \qquad A_0(y,0,D_n)v_0 = f_0(x_n), \qquad 0 < x_n < +\infty,$$

$y = (x', \xi') \in T^*(\partial\Omega)$, $A_0(y, x_n, \xi_n)$ is the principal symbol of $A(x,D)$ in (x', x_n) coordinates, $x' \in \Omega$, $|\xi'| = 1$, and $x_n \in \overline{\mathbf{R}^1_+}$. Assume that $v_0 \in \overset{\circ}{H}_s(\mathbf{R}^1)$ for some s, where $\overset{\circ}{H}_s(\mathbf{R}^1)$ is the subspace of the Sobolev space $H_s(\mathbf{R}^1)$ consisting of functions with supports in $\overline{\mathbf{R}^1_+}$. It can be proven that for all $s \in \mathbf{R}^1$ except a discrete set $\sum_A(x')$ depending on the point $x' \in \partial\Omega$, the equations (3′) define a Fredholm operator $\mathscr{A}_y$ acting from $\overset{\circ}{H}_s(\mathbf{R}^1)$ to $H_{s-\alpha}(\mathbf{R}^1_+)$, where α is the order of A_0. This operator has in general a cokernel as well as a kernel and the dimension of the kernel and of the cokernel may depend on $y \in T^*(\partial\Omega)$. In order to "kill" the kernel of $\mathscr{A}_y$ one should supplement (3) with some boundary conditions as (2). In order to "kill" the cokernel one can add to (3) coboundary operators $K\rho$ acting from the space of functions on the boundary $\partial\Omega$ to the space of functions in Ω. Such operators are also called Poisson operators and they have the form

$$(4) \qquad K\rho = p_\Omega K(x,D)(\rho(x') \otimes \delta(x_n)),$$

where $K(x,D)$ are pseudodifferential operators in $\mathbf{R}^n$, $\delta(x_n)$ is the delta-function, and $\rho(x')$ is a function on $\partial\Omega$. The Fredholm problem for the pseudodifferential operator $A(x,D)$ will have the form

$$(5) \qquad p_\Omega(A(x,D)u_0 + K(x,D)(\rho(x') \otimes \delta(x_n))) = f(x),$$

$$(6) \qquad p_{\partial\Omega} B(x,D)u_0 + E(x',D')\rho(x') = g(x'),$$

where A, K, B are pseudodifferential operators in $\mathbf{R}^n$ and $E(x',D')$ are pseudodifferential operators on $\partial\Omega$. Here f, g are known functions and u_0, ρ are unknown. Formulation of a problem of form (5), (6) and the conditions for which such a problem is Fredholm were established in [**V-E1**, **V-E2**] (see also [**E1**]). In particular it was shown that a Fredholm problem of form (5), (6) exists iff the index of the family of Fredholm operators (3′) is a trivial element of $K(S^*(\partial\Omega))$ (see [**A1**] for the definition of the index of a family of Fredholm operators).

Note that even when $f \in C^\infty(\overline{\Omega})$, the solution $u_0 \in \overset{\circ}{H}_s(\Omega)$ of equation (3) has singularities on $\partial\Omega$.

Independently in [**V-E1**] and [**BdM1**], a class of symbols $A(x,\xi)$ was described such that any solution $u_0 \in L_2(\Omega)$ with $f \in C^\infty(\overline{\Omega})$ belongs to $C^\infty(\overline{\Omega})$. This class of operators was called 'smooth' in [**V-E1**] and was said to 'have the transmission property' in [**BdM1**]. For example all differential operators or operators with even symbols (i.e., $A(x,\xi) = A(x,-\xi)$) have the transmission property. Assume that (3) $A(x,\xi)$ is an elliptic symbol having the transmission property. Without loss of generality one can assume also that the degree of $A(x,\xi)$ is zero and $u_0(x) \in L_2(\Omega)$, i.e., $s = 0$. The transmission property implies that $A_0(y,0,+1) = A_0(y,0,-1)$, where $A_0(y,0,\xi_n)$ is the same as in (3′). Then one can show that

$$(7) \qquad \theta^+ A_0(y,0,D_n) \cdot \theta^+ A_0^{-1}(y,0,D_n) = I + T(y),$$

where $\theta^+(x_n) = 1$ for $x_n > 0$, $\theta^+(x_n) = 0$ for $x_n < 0$, I is the identity operator in $L_2(\mathbf{R}^1_+)$, and $T(y)$ is a Hilbert-Schmidt operator

$$T(y)v(x_n) = \int_0^\infty T(y, x_n, t)v(t)\,dt, \qquad 0 < x_n < +\infty. \tag{8}$$

Moreover the operators

$$\theta^+ A_0(y, 0, D_n) + G(y), \tag{9}$$

where $G(y)$ is a Hilbert-Schmidt operator as in (8), form an algebra. Starting from the algebra (9) Boutet de Monvel [**BdM2**] considered a wider class of boundary problems than (5), (6). They have the form

$$p_\Omega(A(x, D)u_0 + Gu_0 + K(x, D)(\rho(x') \otimes \delta(x_n))) = f(x), \tag{10}$$

$$p_{\partial\Omega}B(x, D)u_0 + E(x', D')\rho(x') = g(x'). \tag{11}$$

The new term G is called the singular Green operator. After "freezing" coefficients and performing the Fourier transform in tangential variables the Green operator has form (8). Boutet de Monvel proved that operators (10), (11) form an algebra, in particular the parametrix for the Fredholm problem (10), (11) also has such form. The parametrix for (5), (6) when $A(x, \xi)$ does not have the transmission property, was constructed in [**E2**]. In the case when $A(x, \xi)$ has the transmission property this parametrix has form (10), (11) with G such that the corresponding Hilbert-Schmidt operator $G(y)$ has a finite rank. Note that when the transmission property does not hold the operators $\theta^+ A_0(y, 0, D_n) + G(y)$ do not form an algebra. An algebra containing $\theta^+ A_0$ was constructed in [**E1**] and it has the form

$$\theta^+(A_0(y, 0, D_n) + \chi M) + G(y), \tag{12}$$

where $\chi(x_n) \in C_0^\infty(\mathbf{R}^1)$, $\chi = 1$ in a neighbourhood of $x_n = 0$ and, M is the so-called Mellin operator. A similar algebra in different terms was described independently by Cordes [**C1**]. Rempel and Schulze in [**R-S1**] applied the Boutet de Monvel construction to the algebra (12) instead of (9) to obtain another form of parametrix in the case when the transmission property is not satisfied.

The book under review is devoted to further development of Boutet de Monvel's calculus of boundary problems for pseudodifferential operators having the transmission property. It starts with a survey of boundary problems (10), (11) including precise descriptions of Green and Poisson operators. The main part of the book is a detailed study of parameter-dependent pseudodifferential operators and boundary value problems. This leads to the parametrix construction for parameter-dependent boundary value problems and its application to the study of the resolvent. The book concludes with various applications: parabolic equations, including heat kernel asymptotics, the index formula for elliptic boundary value problems, complex powers of the operators discussed, spectral asymptotics and singular perturbation problems. Most of the book is based on results of the author, but references and descriptions of related works are also given. The book is well written and it will certainly be useful for everyone interested in boundary value problems and spectral theory.

REFERENCES

[A1] M. F. Atiyah, *K-theory*, Benjamin, Amsterdam, 1967.

[BdM1] Louis Boutet de Monvel, *Comportement d'un opérateur pseudo-differentiel sur une variété à bord.* I, II J. d'Analyse Math. **71** (1966), 241–253; 255–304.

[BdM2] ——, *Boundary problems for pseudodifferential operators*, Acta Math. **126** (1971), 11–51.

[C1] H. O. Cordes, *Pseudo-differential operators on a half line*, J. Math. Mech. **18** (1968/69), 893–908.

[E1] G. I. Eskin, *Boundary value problems for elliptic pseudo-differential equations*, Transl. Math. Monographs, vol. 52, Amer. Math. Soc., Providence, R. I., 1981.

[E2] ——, *Boundary value problems and the parametrix for systems of elliptic pseudodifferential equations*, Trans. Moscow Math. Soc. **28** (1973), 74–115.

[R-S1] S. Rempel and B.-W. Schulze, *Parametrices and boundary symbolic calculus for elliptic boundary value problems without transmission property*, Math. Nachr. **105** (1982), 45–149.

[V-E1] M. I. Vishik and G. I. Eskin, *Convolution equations in a bounded domain*, Russian Math. Surveys **20** (1965), no. 3, 85–151.

[V-E2] ——, *Normally solvable problems for elliptic systems of convolution equations*, Math. USSR-Sb. (1967), 303–330.

GREGORY ESKIN

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 19, Number 1, July 1988

0273-0979/88 $1.00 + $.25 per page

Sobolev spaces of infinite order and differential equations, by Julij A. Dubinskij. B. G. Teubner, Leipzig, and D. Reidel, Dordrecht, Boston, Lancaster, and Tokyo, 1986, 161 pp., $48.00. ISBN 90-277-2147-5

The early fifties, marked by the publication of the book *Applications of functional analysis in mathematical physics* by Sergej L. Sobolev in 1950 (see [**1**]), can be considered the beginnings of a systematical study of function spaces, which soon obtained the name of *Sobolev spaces.* Of course, the foundations of this research had been laid by Sobolev as early as in the thirties by his three papers that appeared in the years 1935–1938. In 1939 Sobolev became the youngest member of the Soviet Academy of Sciences at the age of 31. His approach during this period, which involved among other things the foundations of the theory of distributions, can be seen in his book [**2**] published in 1974. It is not my intention to incite discussions on the question of priority: as is quite frequent in mathematics, there were other mathematicians reflecting on similar problems (J. W. Calkin, Ch. B. Morrey, Jr.), and even the Dirichlet integral can be considered a basis for the theory of Sobolev spaces.

It was discovered that the Sobolev spaces form a very useful tool for introducing modern methods of solution of partial differential equations. Their stormy development was not restricted to the country of their origin (some generalizations and new views of these spaces can be found, e.g., in the books [**3**, **4**] by S. M. Nikol′skij and his colleagues): monographs devoted to the theory of Sobolev spaces appeared also outside the Soviet Union. At first, the