BOOK REVIEWS

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Regular variation, by N. H. Bingham, C. M. Goldie, and J. L. Teugels. Encyclopedia of Mathematics and Its Applications, vol. 27, Cambridge University Press, Cambridge, 1987, xix + 491 pp., $75.00. IBN 0-521-30787-2

Regular variation can be seen initially as an attempt to define the derivative of a real function ϕ at infinity. Write the differential quotient $\{\phi(t+h)-\phi(t)\}/h$ for $h \neq 0$. Now instead of keeping t fixed and letting $h \to 0$, we keep h fixed and let $t \to \infty$. If ϕ is (Borel-) measurable and the limit exists for all $h \neq 0$, then this limit does not depend on h (since the limit of $\phi(t+h)-\phi(t)$ satisfies Cauchy's functional equation). Moreover there exists a differentiable function ϕ_0 such that $\phi_0(t) - \phi(t) \to 0$ $(t \to \infty)$ and

$$\lim_{t\to\infty} \phi_0'(t) = \lim_{t\to\infty} \frac{\phi(t+h)-\phi(t)}{h}.$$

If $f := \exp \circ \phi \circ \log$, then $f\colon \mathbf{R}^+ \to \mathbf{R}^+$ is measurable and the property above translates into

$$\lim_{t\to\infty} \frac{f(tx)}{f(t)} = x^\alpha \quad \text{for all } x > 0; \tag{1}$$

here α is a real parameter. This is the definition of regular variation.

It turns out that many properties that hold identically for power functions, hold asymptotically for functions of regular variation. For example the relation

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{tf(t)} \int_0^t f(s)\,ds &= \lim_{t\to\infty} \int_0^1 \frac{f(tx)}{f(t)}\,dx \\ &= \int_0^1 \lim_{t\to\infty} \frac{f(tx)}{f(t)}\,dx = \int_0^1 x^\alpha\,dx = \frac{1}{1+\alpha}\end{aligned} \tag{2}$$

holds whenever the integrals are finite. In fact relation (2) characterizes regular variation (except for the integrability), i.e., a regularly varying function is precisely a function that is asymptotically of the same order as its average. Relation (2) suggests that there should be some automatic uniformity in relation (1) and indeed this is true on compact x-subsets of $(0,\infty)$. The second equality in (2) also holds with the integration interval $[0,1]$ replaced by $[1,\infty)$. A generalization of both is

$$\lim_{t\to\infty} \frac{1}{tf(t)} \int_0^\infty k(x/t) f(x)\,dx = \lim_{t\to\infty} \int_0^\infty k(x) \frac{f(tx)}{f(t)}\,dx = \int_0^\infty k(x) x^\alpha\,dx. \tag{3}$$

This holds under certain conditions on the kernel, e.g., for $k(x) = e^{-x}$ (Laplace transform), provided the integrals exist, and it leads to the statement that a regularly varying function is precisely a function that is asymptotically (for $t \to \infty$) of the same order as its Laplace transform (for $t \downarrow 0$). It is clear that such a statement is useful in the many situations where the original function cannot be calculated explicitly but the Laplace transform can.

Most of the results mentioned so far are basically due to J. Karamata in a series of papers in 1930–1933. Some other contributors to the theory before 1970 are: S. Aljancić, R. Bojanić (partly publishing in Serbian), N. G. de Bruijn, E. E. Kohlbecker and W. Matuszewska.

An interesting application of this theory in probability theory and a source of new problems became available around 1940 when the convergence of partial sums of independent random variables towards stable laws (W. Doeblin) and the convergence of partial maxima of independent random variables towards extreme-value distributions (B. V. Gnedenko) were characterized. In both problems regular variation of the tail of the probability distribution is necessary and sufficient for convergence in most cases. The connection with Karamata's theory, however, was made much later, essentially with the publication of W. Feller's well-known book on probability theory (1966).

An extra complication occurring in the theory of stable laws (namely when the main parameter is one) and in the theory of extreme value distributions (namely in connection with the double exponential distribution) gave rise to the following generalization of the definition of regular variation: there are real functions $a > 0$ and b such that

$$\lim_{t\to\infty} \frac{f(tx) - b(t)}{a(t)} =: \psi(x) \tag{4}$$

exists for all $x > 0$. If f is measurable, one can prove that ψ is necessarily of the form

$$\psi(x) = \frac{x^\gamma - 1}{\gamma}$$

for some $\gamma \in \mathbf{R}$. Moreover, if (4) holds with $\gamma > 0$, then f is regularly varying (and the other way around) and if (4) holds with $\gamma < 0$, then $\lim_{t\to\infty} f(x) =: f(\infty)$ exists and the function $f(\infty) - f(x)$ is regularly varying. (Here also the converse statement is true as well.) The only novelty is the case $\gamma = 0$ when the right-hand side of (4) should be read as $\log x$. For this class of functions analogues of all previously mentioned statements on regularly varying functions hold.

Another useful generalization of regular variation is obtained for example by requiring that $f\colon \mathbf{R}^+ \to \mathbf{R}^+$ is measurable and by replacing (1) with

$$\limsup_{t\to\infty} \frac{f(tx)}{f(t)} < \infty \tag{5}$$

for all $x > 0$.

I mention some simple (nonrepresentative but easily understood) applications of regularly varying functions.

Differential equations. Relation (2) can be read as follows: if the real function g satisfies the differential equation $g'(x) = a(x) \cdot g(x)$ with a such that $\lim_{x\to\infty} x\, a(x)$ exists in $(0, \infty)$, then g' and hence g is regularly varying. A more complicated result in this direction is the following (due to V. G. Avakumović): if the real function g satisfies the differential equation $g''(x) = u(x)\{g(x)\}^{\lambda}$ with $\lambda > 1$ and u satisfying (1) for some $\alpha > 2$, then

$$\lim_{x\to\infty} g(x)\{x^2 \cdot u(x)\}^{-1/(1-\lambda)}$$

exists and is a simple positive function of λ and α.

Cesàro convergence. A variant of the above-mentioned decomposition of the function ϕ is the following, based on the theory around (4): The function $f(x)$ converges to M in Cesàro mean ($x \to \infty$) if and only if it can be written as $f(x) = a(x) + b(x)$ with the functions a and b satisfying $\lim_{x\to\infty} a(x) = M$ and $\int_1^\infty t^{-1} b(t)\, dt < \infty$ (N. H. Bingham).

Statistics. Suppose (as it is often the case) that a sequence of estimators $\{T_n\}$ for the unknown parameter θ of a probability distribution satisfies: $Z_n := \sqrt{n}(T_n - \theta)$ asymptotically ($n \to \infty$) has a normal distribution. Let h be some real function. One may wonder when the sequence of estimators $\{h(T_n)\}$ for the parameter $h(\theta)$ has a limit distribution ($n \to \infty$). One can write

$$W_n := a_n\{h(T_n) - h(\theta)\} = a_n\{h(\theta + Z_n/\sqrt{n}) - h(\theta)\}$$

with $\{a_n\}$ a suitable sequence of positive scale constants. For example, let h be nondecreasing. One sees that a nondegenerate limit distribution for W_n exists if and only if the functions $f_{\pm}(x) := h(\theta \pm 1/x) - h(\theta)$ are regularly varying and suitably balanced.

The book under review is a comprehensive account of the theory of regular variation, its generalizations and its applications. The book is well conceived: important statements receive full emphasis, but for the interested reader there is a wealth of side remarks, references to various parts of mathematics and variants. Everything is presented in a precise but pleasant way. In most parts of the book there is a good balance between generality and attractiveness (this cannot always be said about the older book by E. Seneta). One third of the book is devoted to applications (in analytic number theory, complex analysis and probability theory). If one does not consider Tauberian theorems as an integral part of the theory, more than half of the book consists of applications. Some of the newer theory could not be covered. At one place I find the set-up of the book unsatisfactory. First the theory around relation (1) is developed. Next inequalities of, e.g., type (5) are considered. So far so good. But when the authors start with generalizations of type (4), they first develop the theory around inequalities connected with (4) and only after these they consider relation (4) itself. It would have been preferable to develop the theory around (4) in the same order as they theory around (1).

The book by Bingham, Goldie and Teugels presents a broad and variegated area of knowledge in an orderly, accessible and yet impressive way. An opera of real analysis, so to speak.

REFERENCES

E. Seneta (1976), *Regularly varying functions*, Lecture Notes in Math., vol. 508, Springer-Verlag, Berlin and New York.

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BULLETIN (New Series) OF THE
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Ill-posed problems of mathematical physics and analysis, by M. M. Lavrent'ev, V. G. Romanov, and S. P. Shishatskiĭ; translated by J. R. Schulenberger. Translations of Mathematical Monographs, vol. 64, American Mathematical Society, Providence, R. I., 1986, vi + 290 pp., $98.00. ISBN 0-2818-4517-9

As is well known, a problem is said to be well-posed in the sense of Hadamard when a unique solution exists and depends continuously upon the data. The definition is made precise by stipulating not only the function spaces in which the solution and data are to lie but also the measures and notion of continuity. A problem that is not well-posed is said to be ill-posed.

Although nineteenth-century mathematicians contributed to the early study of ill-posed problems, it is generally agreed that the subject came to prominence only after Hadamard had formulated his well-known definition. Unfortunately, he developed an adverse view of the subject which, on becoming widely accepted, had the effect of inhibiting further study. His objections were grounded in his celebrated counterexample of the Cauchy problem for Laplace's equation. In order for there to be global existence of the solution Hadamard demonstrated that the Cauchy data must satisfy a certain compatibility relation but even in the unlikely event of the relation being satisfied he further showed that the solution in general does not depend continuously on the data. Such behaviour convinced Hadamard that ill-posed problems lacked physical relevance and hence should be ignored. This became the prevailing attitude, and consequently, in partial differential equations at least, activity became confined to the standard initial boundary value problems. It was only the growing insistence for a precise theoretical understanding from the applied sciences, principally geophysics and computing, that rekindled mathematical interest.

It is worth considering briefly why ill-posed problems are of practical importance and hence merit detailed study. Take, for example, the simple Dirichlet problem for a linear elliptic homogeneous differential equation. Conditions are known guaranteeing that the solution exists, is unique and depends continuously upon the Dirichlet data, i.e., the problem is well-posed. These conditions include the requirement that the solution be specified in a suitable sense at all points of the boundary of the region of definition. Yet, rarely, if ever, can this specification be completely achieved in practice. Measuring devices record only approximate values and in any case are able to measure