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ON THE LOCAL SEVERI PROBLEM

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Introduction. We study plane curves with singularities. Let $\mathbf{P}^N$ be the projective space parametrizing plane curves of degree n ($N = n(n+3)/2$). Let $V(n,g) \subset \mathbf{P}^N$ be the locus of reduced irreducible plane curves of degree n and (geometric) genus g, and $l \subset \mathbf{P}^2$ a fixed line. Following Zariski [**7**], we consider the subvariety $Z(n,g) \subset \overline{V(n,g)}$ of curves which contain l as a component. The purpose of this note is to study $Z(n,g)$ and prove the following

THEOREM. *Let $\mathscr{E}(n,g)$ be a branch of $\overline{V(n,g)}$ through a point of $Z(n,g)$ corresponding to a reduced curve. Then the general members of $\mathscr{E}(n,g) \cap Z(n,g)$ have only nodes as singularities.*

It is well known (cf. Severi [**5**, §11]) that this Theorem implies the following fundamental result of Harris.

COROLLARY (HARRIS [**3**]). *$V(n,g)$ is irreducible.*

In the case when $L \in \mathscr{E}(n,g) \cap Z(n,g)$ is a union of n distinct lines passing through a point, our theorem is a realization of Severi's attempt to prove that L can be regenerated to a reducible nodal curve of $\mathscr{E}(n,g)$ [**5**, §11, p. 344]. The idea of using decreasing induction on g and equations of curves in the proof was suggested in Zariski [**7**]. On the other hand, Harris [**3**] and Ran [**4**] use the degeneration method in their treatment of plane curves.

Proof of Theorem. We set $d = (n-1)(n-2)/2 - g$ and $\nu(n,d) = \dim V(n,g) = 3n + g - 1$ ([**5**, §11], [**6**]). Let $\Sigma_{n,d} \subset \mathbf{P}^N \times \mathrm{Sym}^d(\mathbf{P}^2)$ be the closure of the locus of irreducible curves of degree n with d nodes and no other singularities, and π_N the projection to $\mathbf{P}^N$. Given a pair consisting of a reduced curve $E \in \overline{V(n,g)}$ and a branch of $\overline{V(n,g)}$ through the curve, one can define, via π_N, an element of $\mathrm{Sym}^d(\mathbf{P}^2)$, called the cycle of assigned singularities of the pair. Our basic tool is the dimension-theoretic characterization of maximal families of nodal curves by Arbarello and Cornalba [**1**] and Zariski [**6**] and its generalization by Harris [**3**, Proposition 2.1].

Let C be a general member of $\mathscr{E}(n,g) \cap Z(n,g)$. We will prove that C is nodal and all its unassigned nodes lie on l for every choice of a branch of $\mathscr{E}(n,g)$ through C.

LEMMA. *For $d \leq 3$, $\Sigma_{n,d}$ is irreducible and unibranch.*

PROOF OF THE LEMMA. Let Σ', $\Sigma'' \subset \Sigma_{n,d}$ be components such that a general member of Σ' has d nodes in general position. A dimension count

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shows that a general member of $\Sigma' \cap \Sigma''$ is a reduced curve with d assigned singular points. Therefore $\Sigma_{n,d} = \Sigma'$. The second assertion follows from the unibranchness of $\mathrm{Sym}^d(\mathbf{P}^2)$.

Step 1. Let $d \leq 3$. We regenerate C to a nodal curve $F = l + F'$ having the same number of irreducible components and the same genus as C. We take a branch of $\mathscr{E}(n,g)$ through C and consider the corresponding cycle of assigned singularities $\sum d_t P_t$. For every t, we choose d_t assigned nodes of F in the vicinity of P_t. The curve F with the $d = \sum d_t$ assigned nodes determines a branch of $\overline{V(n,g)}$ through C. By the lemma, it coincides with the branch of $\mathscr{E}(n,g)$, chosen above.

Step 2. We assume $d \geq 4$ and the theorem is true for smaller d's. If $C = l + C'$ has no assigned singularities on l (for a branch of $\mathscr{E}(n,g)$), then C' is moving in the family of dimension $\leq \nu(n-1,d) = \nu(n,d) - n - 1$. Since $Z(n,g)$ is defined by $n+1$ equations in $\mathscr{E}(n,g)$ [**7**, p. 470], the inequality is, in fact, equality and C is nodal.

Step 3. We now assume that C has assigned singularities on l. Let $f(X,Y,Z) = \Sigma a_{jk} X^j Y^k Z^{n-j-k}$ be an equation of a curve of $\mathscr{E}(n,g)$. We have chosen our coordinate system in $\mathbf{P}^2$ so that $l = \{X = 0\}$ and all the singularities of the curves of $\mathscr{E}(n,g)$ lie in $\mathbf{P}^2 \backslash \{Z = 0\}$. Moreover, the following constructions take place in a neighborhood of a general member D of $\mathscr{E}(n,g) \cap Z(n,g) \cap (a_{00} = a_{10} = a_{01} = 0\}$. We assume D is a specialization of C, and it has an *assigned* singularity at (0:0:1). By abuse of notation we denote by $\mathscr{E}(n,g)$ a fixed new branch of $\overline{V(n,g)}$ through D, contained in the original branch.

Let $\mathscr{A}_g^{\,0} \subset \mathscr{E}(n,g)$ be the subfamily of curves having a node at (0:0:1). It has codimension 2 in $\mathscr{E}(n,g)$ and its general members are irreducible nodal curves with d nodes. For $i \geq 1$, the components of $\mathscr{A}_g^{\,i} = \mathscr{A}_g^{\,i-1} \cap \{a_{0n-i+1} = 0\}$ have codimension $\leq i+2$ in $\mathscr{E}(n,g)$ and consist of curves having intersection multiplicity at least i with l at (0:1:0). If a general member of $\mathscr{A}_g^{\,i}$ does not contain l as a component, then a dimension count shows that it has intersection multiplicity i with l at (0:1:0) and only d singular points which are nodes: we blow up (0:0:1) and i times (0:1:0) in the direction of l, and apply [**3**, Proposition 2.1].

Step 4. Let E be a general member of $\mathscr{E}(n,g)$ and $Q \in E$ a node distinct from (0:0:1). Moreover, if D has an assigned (with respect to $\mathscr{E}(n,g)$) singular point outside l, then we assume Q tends to this point; the second choice is a node $Q \in E$ which does not tend to (0:0:1). Let $\mathscr{E}(n,g+1) \subset \overline{V(n,g+1)}$ ($\mathscr{E}(n,g) \subset \mathscr{E}(n,g+1)$) be the branch through D obtained by considering Q as virtually nonexistent. We can define, as above, the subfamilies $\mathscr{A}_{g+1}^{i}$ of $\mathscr{E}(n,g+1)$. Let $m+1$ be the first integer such that a general member of $\mathscr{A}_g^{\,m+1}$ contains l as a component.

Case 1. The general members of $\mathscr{A}_{g+1}^{m+1}$ do not contain l as a component. We get $\dim \mathscr{A}_{g+1}^{m+2} = \dim \mathscr{A}_g^{\,m+1}$. By the induction hypothesis we get that D is nodal.

Case 2. A general member F of $\mathscr{A}_{g+1}^{m+1}$ contains l as a component. Then $F = l + F'$ is nodal. A dimension count shows that $D = l + D'$ can have at most one non-nodal singularity. Moreover, if $g(F') \neq g(D')$ then D is nodal. If D is not nodal and $g(F') = g(D')$, then the singular points of F and D are on l. Hence F' is smooth and D has one tacnode. We can assume the node Q tends to a node $Q^* \in D$. Let $\mathscr{H} \subset \Sigma_{n,1}$ be the branch through (D, Q^*). By [**2**, Exp. XIII, §2],

$$\mathscr{B} = \mathscr{E}(n, g+1) \cap \pi_N(\mathscr{H})$$

is 1-connected. If $\mathscr{B} = \mathscr{E}(n, g)$, we are done. If $\mathscr{B} \neq \mathscr{E}(n, g)$, we choose a component $\mathscr{C}$ of $\mathscr{B}$ such that a component $\mathscr{U}$ of $\mathscr{C} \cap \mathscr{E}(n, g)$ has dimension $\nu(n, d+1)$. Let G be a general member of $\mathscr{U}$. By the deformation theory, we get $g(G) \leq g - 1$. Therefore G is nodal with $d+1$ nodes, two of which tend to the tacnode. By intersecting $\mathscr{U}$ with the branches of $\pi_N(\Sigma_{n,1})$ corresponding to the unassigned nodes of D, we derive that D is not a general curve.

REMARKS. A dimension count shows that the number of unassigned singularities of C is equal to m.

As in the lemma, for $d \leq n(n+3)/6$ and $(n, d) \neq (6, 9)$, the unibranchness of $\Sigma_{n,d}$ follows from the irreduciblity. One can give another proof that $\Sigma_{n,d}$ is irreducible in that range using the following general result (see a conjecture in [**7**, p. 479]): Let E be a general member of an irreducible subfamily S of $\overline{V(n, g)}$ $(0 \leq g \leq (n-1)(n-2)/2)$. If $\dim S \geq \nu(n, d) - 3$, then E is reduced. If S consists of nonreduced curves and has the maximal dimension, then a general member of S is a union of an irreducible nodal curve and a general double line.

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