

THE NUMBER OF EQUATIONS NEEDED TO DEFINE AN ALGEBRAIC SET

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Let B be a commutative Noetherian ring, $X = \text{Spec } B$ the associated affine scheme, $I \subset B$ an ideal and $V = V(I) \subset X$ the closed subset defined by I .

DEFINITION. Elements $f_1, \dots, f_s \in I$ define V set-theoretically (equivalently, V is defined set-theoretically by s equations $f_1 = 0, f_2 = 0, \dots, f_s = 0$) if $\sqrt{(f_1, \dots, f_s)} = \sqrt{I}$.

Hilbert's Nullstellensatz implies that in the case when B is a finitely generated algebra over an algebraically closed field k , this definition agrees with the usual one, i.e., all f_1, \dots, f_s vanish at a k -rational point if and only if it belongs to V . In the sequel "defined" always means "defined set-theoretically".

The question we are dealing with here concerns the minimum number of equations needed to define a given $V \subset X$. A classical result that goes back to L. Kronecker [Kr] says that if B is n -dimensional, then $n + 1$ equations would suffice for every $V \subset X$. Our first theorem describes those $V \subset X$ which can be defined by n equations.

THEOREM A. Let k be an algebraically closed field, X a smooth affine n -dimensional variety over k with coordinate ring B , and $V = V' \cup P_1 \cup P_2 \cup \dots \cup P_r$ an algebraic subset of $X = \text{Spec } B$, where V' is the union of irreducible components of positive dimensions and P_1, P_2, \dots, P_r some isolated closed points (which do not belong to V'). Then V can be defined by n equations if and only if one of the following conditions holds.

(i) $r = 0$, i.e., V consists only of irreducible components of positive dimension.

(ii) V' is empty, i.e., V consists only of closed points and there exist positive integers n_1, n_2, \dots, n_r such that $n_1 P_1 + n_2 P_2 + \dots + n_r P_r = 0$ in $A_0(X)$.

(iii) V' is nonempty, $r \geq 1$ and there exist positive integers n_1, n_2, \dots, n_r such that $n_1 P_1 + n_2 P_2 + \dots + n_r P_r$ belongs to the image of the natural map $A_0(V') \rightarrow A_0(X)$ induced by the inclusion $V' \rightarrow X$.

Here $A_0(\cdot)$ stands for the group of zero-cycles modulo rational equivalence [Fu].

SKETCH OF PROOF. Our proof consists of three steps. In Step 1 we construct an ideal $I \subset B$ such that \sqrt{I} is the defining ideal of V , and in addition I has some other special properties. In Step 2 we, in a special way, pick some ideals Q_1, \dots, Q_n such that $\sqrt{Q_i}$ is a maximal ideal containing I for each i and J/J^2 is n -generated, where $J = I \cap Q_1 \cap \dots \cap Q_n$. In Step 3 we

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prove that the 0-dimensional Segre class of J equals 0, so by [Mu2, Theorem 2] and the Suslin cancellation theorem [Su], J is n -generated. \square

If B is not assumed to be regular, the result of Theorem A need not hold. In fact, for every $n \geq 0$, we have constructed an example of a finitely generated n -dimensional algebra B over a suitable algebraically closed field, such that its singular locus consists of just one closed point and for every d between 1 and $n - 1$ it contains a d -dimensional subvariety which cannot be defined by n equations.

Storch [St] and Eisenbud-Evans [EE] proved that every algebraic set in \mathbf{A}_k^n can be defined by n equations, while Cowsik-Nori [CN] proved that every curve in \mathbf{A}_k^n , where $\text{char } k = p > 0$, can be defined by $n - 1$ equations. Our next theorem sharpens these results.

THEOREM B. *If $\text{char } k = p > 0$, then every algebraic set $V \subset \mathbf{A}_k^n$ consisting only of irreducible components of positive dimensions can be defined by $n - 1$ equations.*

SKETCH OF PROOF. The main idea of the proof is contained in the following lemma.

LEMMA. *Let $f_1, \dots, f_r, g_1, \dots, g_r \in I$ and $a \in B$ such that*

- (i) f_1, \dots, f_r generate I_a up to radical.
- (ii) g_1, \dots, g_r generate $(I + (a))/(a) \subset B/(a)$ up to radical. Then there exist $r + 1$ elements h_1, \dots, h_{r+1} which generate I up to radical.

Now assume all irreducible components of V have dimension $d \geq 1$ and use induction on d , the case $d = 1$ being settled by [CN]. Let $I = I(V) \subset B = k[x_1, \dots, x_n]$ be the defining ideal of V . By a change of variables we can assume that $k[x_n]$ has zero intersection with every minimal prime over-ideal of I . By induction the extension of I in $k(x_n)[x_1, \dots, x_{n-1}]$ can be generated up to radical by $n - 2$ elements. Thus there exists a square-free polynomial $a(x_n)$ such that $I_{a(x_n)}$ can be generated up to radical by $n - 2$ elements. Since $k[x_n]/a(x_n)$ is a product of fields, by induction $(I + (a))/(a) \subset (k[x_n]/a(x_n))[x_1, \dots, x_{n-1}]$ also can be generated up to radical by $n - 2$ elements. Now by the lemma, I can be generated up to radical by $n - 1$ elements.

If V is not equidimensional, the proof is considerably harder, but the main idea remains the same. \square

The rest of this paper is devoted to a question of M. P. Murthy [Mu1], who asked whether every locally complete intersection (l.c.i.) subscheme $V \subset \mathbf{A}_k^n$ is a set-theoretic complete intersection. The answer is known to be positive if $\dim V = 1$ [Fe, Sz, Bo, MK].

We generalize this result as follows:

THEOREM C. *Every l.c.i. subscheme $V \subset \mathbf{A}_k^n$ of constant positive dimension can be defined by $n - 1$ equations.*

A proof of this theorem is similar to that of the equidimensional case of Theorem B. \square

Moreover, we have obtained the following:

PROPOSITION. *Let k be an algebraically closed field of characteristic $p > 0$, I a l.c.i. of equidimension d in $B = k[x_1, \dots, x_n]$ and $n \geq 3d$. Set $A = B/\sqrt{I}$. Then there exists a l.c.i. $J \subset I$ such that $\sqrt{J} = \sqrt{I}$ and $J/J \cdot \sqrt{I}$, the reduced conormal module of J , is isomorphic to $P \oplus A^{n-2d+1}$, where P is a projective A -module of rank $d - 1$ with trivial determinant.*

SKETCH OF PROOF. By the Ferrand construction [Fe, p. 24; Va, p. 89] we are reduced to the case when the determinant of the conormal module of I is trivial. Let $c_d \in F^d(\text{Spec } A)$ be the top Chern class of the reduced conormal module of I , where $F^d(\text{Spec } A)$ is the d th component of the Grothendieck γ -filtration [FL, p. 48]. Since $F^d(\text{Spec } A)$ is divisible, we can write $c_d = (1-p^d)c$. Let Q be a projective A -module of rank d such that $Q - A^d \in F^d(\text{Spec } A)$ and the top Chern class of Q equals c . Set $I/I \cdot \sqrt{I} \approx M \oplus Q$, where M is projective of rank $n - 2d \geq d$. We show that there exists a l.c.i. $J_1 \subset I$ such that $\sqrt{J_1} = \sqrt{I}$ and the reduced conormal module of J_1 is $M \oplus F(Q)$, where $F(Q)$ is the Frobenius of Q . It is straightforward to compute that the top Chern class of $M \oplus F(Q)$ is zero. Taking the iterated Frobenius of J_1 , if necessary, we obtain, by [Mu2, Theorem 5] an l.c.i. J with required properties. \square

THEOREM D. *Let k be any algebraically closed field of characteristic $p > 0$ and $V \subset \mathbf{A}_k^n$ a locally complete intersection subscheme of constant dimension d , such that $2 \leq d \leq n - 4$. Then V can be defined by $n - 2$ equations. In particular, every locally complete intersection surface in \mathbf{A}_k^n , where $n \geq 6$, is a set-theoretic complete intersection.*

PROOF. By induction, using the lemma we reduce to the case $d = \dim V = 2$. By the proposition there exists l.c.i. $J \subset I$ such that $\sqrt{J} = \sqrt{I}$ and the reduced conormal module of J is free. Now by [MK, Theorem 5] J is $(n - 2)$ -generated. \square

THEOREM E. *Let k be any algebraically closed field of characteristic 2 and $V \subset \mathbf{A}_k^n$ a locally complete intersection subscheme of constant dimension d , such that $3 \leq d \leq n - 6$. Then V can be defined by $n - 3$ equations. In particular, every locally complete intersection threefold in \mathbf{A}_k^n , where $n \geq 9$, is a set-theoretic complete intersection.*

SKETCH OF PROOF. By an inductive argument with the help of the lemma we are reduced to the case $d = 3$. By the proposition, we get $J \subset I$ such that $\sqrt{J} = \sqrt{I}$ and the reduced conormal module of J is $P \oplus A^{n-5}$, where P is projective of rank 2 with trivial determinant. By a special argument in characteristic 2, we then obtain a new l.c.i. ideal $J_1 \subset J$ such that $\sqrt{J_1} = \sqrt{I}$ and the reduced conormal module of J_1 is free. Now we are done by [MK, Theorem 5]. \square

Complete proofs will appear elsewhere.

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