RESEARCH ANNOUNCEMENTS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 19, Number 1, July 1988

BUCHSBAUM SUBVARIETIES OF CODIMENSION 2 IN $\mathbf{P}^n$

MEI-CHU CHANG

Over the past several years there has been considerable activity in the topics of Buchsbaum rings and arithmetically Buchsbaum projective curves, some of which was described in the book [**3**]. Recall that a local (resp. graded) ring A is Buchsbaum if the difference $l_A(A/\mathbf{x}A) - e(\mathbf{x}, A)$ is independent of the (homogeneous) system of parameters $\mathbf{x}$, where l_A and e are the length and multiplicity respectively. A subvariety Y in $\mathbf{P}^n$ is called arithmetically Buchsbaum if its homogeneous coordinate ring $\bigoplus H^0(\mathcal{O}_{\mathbf{P}^n}(k))/\bigoplus H^0(\mathcal{I}_Y(k))$ is Buchsbaum. This in particular implies

(1) *the multiplication map* $H^p(\mathcal{I}_Y(k)) \xrightarrow{x} H^p(\mathcal{I}_Y(k+1))$ *is* 0 *for all* $x \in H^0(\mathcal{O}_{\mathbf{P}^n}(1))$ *and* $1 \leq p \leq \dim Y$.

Conversely, if we have (1) and

(2) *whenever* $H^p(\mathcal{I}_Y(k)) \neq 0 \neq H^q(\mathcal{I}_Y(h))$ *for* $1 \leq p < q \leq \dim Y$, *then* $(p+k) - (q+h) \neq 1$,

then Y is arithmetically Buchsbaum (cf. [**3**]). (Note that (2) is vacuous if $n = 3$.)

The purpose of this note is to announce a structure theorem for arithmetically Buchsbaum subvarieties of codimension 2 satisfying (2). We also give several applications of this theorem, including some inequalities among the number and the degree of the generators of the homogeneous ideal I_Y and the degree of Y, nonexistence of nonsingular codimension 2 Buchsbaum subvarieties satisfying (2), in $\mathbf{P}^n$ for $n \geq 6$, and classification of the nonsingular ones for $n \leq 5$.

Received by the editors July 28, 1987 and, in revised form, November 6, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 14F05, 14H45, 14J25, 14J30, 14M05, 14M07; Secondary 14F10, 14M12.
Research partially supported by the NSF.

Our structure theorem is as follows:

THEOREM 1. *If Y is a locally Cohen-Macaulay, codimension 2 subvariety in $\mathbf{P}^n$, having properties (1) and (2), then the ideal sheaf I_Y admits the following exact sequence, called an Ω-resolution:*

$$0 \to \bigoplus_i \mathcal{O}(-a_i) \xrightarrow{\psi} \bigoplus_j l_j \Omega^{p_j}(-k_j) \bigoplus_s \mathcal{O}(-c_s) \to I_Y \to 0, \tag{3}$$

where $h^{p_j}(I_Y(k_j)) = l_j$ are the only nonzero cohomologies for $1 \leq p_j \leq n-2$.

REMARK 1.1. From (3) one can easily construct a free (maybe nonminimal) resolution of I_Y

$$\begin{aligned} 0 &\to \bigoplus_j l_j \tbinom{n+1}{p_j+n} \mathcal{O}(-p_j - n - k_j) \to \cdots \to \bigoplus_j l_j \tbinom{n+1}{p_j+3} \mathcal{O}(-p_j - 3 - k_j) \\ &\to \bigoplus_j l_j \tbinom{n+1}{p_j+2} \mathcal{O}(-p_j - 2 - k_j) \bigoplus \mathcal{O}(-a_i) \\ &\to \bigoplus_j l_j \tbinom{n+1}{p_j+1} \mathcal{O}(-p_j - 1 - k_j) \oplus \mathcal{O}(-c_s) \to I_Y \to 0. \end{aligned}$$

REMARK 1.2. Note that property (1) is not inherited by a hyperplane section of Y; however, together (1) + (2) are. Moreover, to see that property (2) is essential for the theorem, one can easily construct an example by taking Y to be a dependency locus of the dual of a vector bundle which is the quotient of $\Omega^2(3)$ on $\mathbf{P}^4$.

Motivated by Theorem 1, we make the following

DEFINITION 2. A locally Cohen-Macaulay subvariety Y in $\mathbf{P}^n$ is called geometrically Buchsbaum, if Y has an Ω-resolution (3).

QUESTION 2.1. What is the relation between the arithmetically Buchsbaum and geometrically Buchsbaum properties? It is easy to see that the latter implies the former. From Theorem 1, we know that the latter is also between properties (1)+(2) and (1). In $\mathbf{P}^3$, they are the same.

In the rest of this note, we will describe what we can say about geometrically Buchsbaum subvarieties of codimension 2. The following is an easy observation.

PROPOSITION 3. *Let Y be a geometrically Buchsbaum subvariety of codimension 2 in $\mathbf{P}^n$. If Y is the zero set of a rank 2 vector bundle E on $\mathbf{P}^n$, then E is the null-correlation bundle on $\mathbf{P}^3$.*

COROLLARY 3.1. *There is no nonsingular geometrically Buchsbaum subvariety of codimension 2 in $\mathbf{P}^n$ for $n \geq 6$.*

The following result allows us to construct many nonsingular geometrically Buchsbaum subvarieties of codimension 2 in $\mathbf{P}^n$ for $n < 6$.

THEOREM 4. *Suppose Y has an Ω-resolution (3) and ψ is sufficiently general, and $a_1 \leq \cdots \leq a_r$. We replace each copy of $\Omega^p(-k)$ by $\binom{n}{p}$ copies of $\mathcal{O}(-p-1-k)$, and denote the new direct sum of line bundles by $\bigoplus \mathcal{O}(-b_i)$ with $b_1 \leq \cdots \leq b_{r+1}$. Then*

(a) *If $a_i \geq b_{i+1}$ for all i, then Y is of codimension 2.*

(b) *If* $a_i \geq b_{i+\alpha}$ *for all* i, *then* Y *is nonsingular except for a subset of codimension* $\geq \min\{2\alpha - 1, 4\}$.

REMARK 4.1. We believe that in (a) and (b) 'if' can be replaced by 'if and only if', when $n < 6$.

Some trivial consequences in $\mathbf{P}^3$ are, e.g.,

(i) If a Buchsbaum curve Y in $\mathbf{P}^3$ with only one nonzero $h^1(I_Y(k)) = n$ (i.e. in the class L_n, cf. [**1**]) is nonsingular and of maximal rank, then

$$2n^2 + 2ln + l(l-1)/2 \leq \deg Y \leq 2n^2 + 2ln + l^2,$$

where $l = k + 2 - 2n$. Conversely, for any d in the range above, there is a Y, with all the properties above, of degree d. (*Remark*: $k + 1 = l + 2n - 1$ is σ, the length of the numerical character of Y: cf. [**1**].)

(ii) Given a set of ordered pairs of nonnegative integers $\{(k_i, n_i) \mid k_1 \leq \cdots \leq k_m\}$, a necessary and sufficient condition for the existence of a (nonsingular) Buchsbaum curve Y in $\mathbf{P}^3$ having $h^1(I_Y(k_i)) = n_i$ as the only nonzero intermediate cohomologies is $k_1 \geq 2\sum_1^m n_i - 2$ (resp. $2k_1 - k_m \geq 2\sum_1^m n_i - 2$). Similar conclusions can be drawn for $\mathbf{P}^n$ for $n > 3$.

From the Ω-resolution, we can easily compute Chern numbers and other numerical characters of Y, and hence study the abstract model as well as the embedding of Y in $\mathbf{P}^4$ and $\mathbf{P}^5$. All Y are of general type except a finite list, which we have classified.

Another use of the Ω-resolution is to obtain various inequalities involving the invariants of $Y \subset \mathbf{P}^n$. Some examples of what we can get are (cf. [**2**]):

(iii)
$$\mu \leq \frac{(\sqrt{8(n-1)^2 d + (n-3)^2} + n - 3)n}{4(n-1)} + 1,$$
$$N \leq \frac{(\sqrt{8(n-1)^2 d + (n-3)^2} + n - 3)}{2(n-1)^2}$$

where μ is the number of generators of $I(Y)$, $d = \deg Y$, and

$$N = \sum_j h^{p_j}(I_Y(k_j)).$$

The bounds are obtained when, in resolution (3), $p_j = 1$, $a_i = k_j + 2$ for all i and j, and there are no line bundles $\mathcal{O}(-c_s)$ in the middle.

(iv) $a \leq d - (n^2 - 3n + 4)/2$, where $d = \deg Y$ and $a =$ the maximal degree of minimal generators of $I(Y)$, assuming

$$a \geq \binom{n-1}{[\frac{n-1}{2}]} \Big/ 2$$

for simplicity. This bound is obtained for Y with the resolution

$$0 \to \mathcal{O}(-2n+1) \to \ldots \to \binom{n+1}{3}\mathcal{O}(-n-1) \oplus \mathcal{O}(-a-1)$$
$$\to \frac{n^2 - n + 2}{2}\mathcal{O}(-n) \oplus \mathcal{O}(-a) \to I_Y \to 0.$$

REMARK. If Y is projectively Cohen-Macaulay, then $\mu \leq \sqrt{2d+1/4}+\frac{1}{2}$ and $a \leq d-1$.

Other bounds of a (as defined in (iv)) in terms of $\deg Y$ and $\sum h^i(I_Y(k))$ (resp. the minimal degree of the generators) can be computed in a similar manner, as can the dimensions of the families.

ACKNOWLEDGMENT. This announcement was written while the author was staying at the Mittag-Leffler Institute during the spring of 1987, also (morally) supported by the NSF. The author thanks Z. Ran and K. Ranestad for helpful discussions.

ADDED IN PROOF. In response to Question 2.1, it is now known that the notions of arithmetically Buchsbaum and geometrically Buchsbaum are equivalent in codimension 2. See a forthcoming paper by the author.

REFERENCES

1. G. Bolondi and J. Migliore, *Maximal rank curves in the liaison class* k^n (preprint).

2. P. Maroscia, J. Stückrad, and W. Vogel, *Upper bounds for the degrees of the equations defining locally Cohen-Macaulay schemes*, Math. Ann. **277** (1987), 53–65.

3. J. Stückrad and W. Vogel, *Buchsbaum rings and applications*, Springer-Verlag, 1986.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, LOS ANGELES, CALIFORNIA 90024

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, RIVERSIDE, CALIFORNIA 92521