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Global stability of dynamical systems, by Michael Shub, with the collaboration of Albert Fathi and Remi Langevin; translated by Joseph Christy. Springer-Verlag, New York, Berlin, Heidelberg, London, Paris, Tokyo, xii + 150 pp., $32.00. ISBN 0-387-96295-6

In 1967 Smale wrote an important survey article [1][1] outlining a program for the study of smooth dynamical systems on manifolds. The program has had both widespread influence and some important successes. In this review, I shall try to present a rough idea of the program in order to show how the book under review fits in.

To begin with let us recall that Smale won the Fields medal for proving the Poincaré conjecture in dimensions greater than 5. The technique he used was that of Morse theory, where one takes a function defined on a manifold M and analyzes the topology of the manifold using the gradient flow of this function. Roughly speaking, one has a decomposition of the manifold as a union of unstable manifolds $W^u(p)$ where p ranges over the critical points of the function and $W^u(p)$ is the unstable manifold of the critical point p:

$$M = \bigcup_{p\in\Omega} W^u(p),$$

where

$$\Omega = \{p \in M : f^t(p) = p \text{ for all } t\}$$

and

$$W^u(p) = \{x \in M : \lim_{t\to-\infty} d(f^t(x), f^t(p)) = 0\}$$

with f^t the flow and d a suitable metric on the manifold M. After this spectacular success he set about studying the dynamical system for its own sake rather than as a tool to get at the topology of M. Of course, he knew that the above decomposition would be tractable only if the gradient function satisfied some nondegeneracy conditions (for example that its critical points be nondegenerate) and so quite naturally, he wanted to first study the "generic" case.

Now of course any vector field (i.e., ordinary differential equation) on M will give a flow, but it is easy to see that in general we don't have such a simple decomposition. The reason is that (nongradient) vector fields can have periodic orbits and no two distinct points in the same periodic orbit can lie in the same unstable manifold. Smale was led to conjecture that suitable nondegeneracy hypotheses would lead to a decomposition

$$(*) \qquad M = \bigcup_i W^u(\Omega_i)$$

[1]The interested reader should study the reprint [2] of Smale's paper rather than the original version since the reprint corrects some errors and includes an appendix which describes subsequent work in the area.

where for any set $\Lambda \subset M$ we define

$$W^u(\Lambda) = \bigcup_{p \in \Lambda} W^u(p)$$

and where $\Omega_1, \Omega_2, \ldots, \Omega_n$ are the critical elements of the flow f^t; i.e., the critical points and periodic orbits. (In particular, the conjecture said that there were only finitely many critical elements.)

The conjecture was shot down almost immediately. Levinson pointed out that his work [**3**] extending work of Cartwright and Littlewood concerned very general examples of generic differential equations with infinitely many periodic orbits and that indeed such examples were already known to Poincaré in the equations of classical mechanics. (Of course, the harmonic oscillator has infinitely many periodic solutions but this example is degenerate in the sense that small perturbations of it can be constructed which have no periodic solutions except for the rest point.) Other examples were found in other classical subjects. For example, the geodesic flow on a surface of constant negative curvature can be more or less explicitly calculated and shown to have infinitely many distinct periodic solutions. The important work of Anosov [**4**] extended this to the geodesic flow of any compact manifold of strictly negative curvature (constant or not). More importantly, Anosov showed that any such flow was *structurally stable* in that a small perturbation of it preserved the topology of the flow. Perhaps the most dramatic example was given by Thom who remarked that the diffeomorphism (discrete time dynamical system) $f\colon T^2 \to T^2$ of the torus given by

$$f(x, y) = (2x + y, x + y) \mod \mathbf{Z}^2$$

had periodic orbits of every (integer) period and that since this could be proved using the Lefschetz fixed point theorem it would be true of any small perturbation of f. Smale himself produced his famous horseshoe example which showed that infinitely many periodic orbits could be expected in any system having a "transversal homoclinic point".

This was the state of affairs at the time of Smale's 1967 paper. That paper introduced a key definition which these days cay be expressed in the form

The chain recurrent set of f is hyperbolic.

It also contained two important conjectures:

A system is Ω-stable iff its chain recurrent set is hyperbolic;

and

A system is structurally stable iff its chain recurrent set is hyperbolic and it satisfies the strong transversality condition.

(The book under review defines these terms precisely.) The *if* directions of these conjectures were proved within the next five years (see Smale's reprint [**2**] for references) and it appears that the converse directions will also be announced soon. For our purposes in this review it is enough to note that the hyperbolicity conditions generalize the conditions Anosov defined in his work, the strong transversality condition generalizes the condition that all homoclinic points be transverse, and the stability conditions say that important topological properties of the system are unchanged by C^1 small perturbations of it.

The single most important result in Smale's 1967 paper is the so-called *spectral decomposition theorem* which proves a decomposition of the form $(*)$ where the sets Ω_i are the components of a certain decomposition of the chain recurrent set. The local behavior of the system near each Ω_i can be understood in terms not much different from the local behavior in the gradient case and so in a certain sense corrects Smale's original naive conjecture.

Now we are ready to discuss what appears in the book under review. First, we should point out that the book deals exclusively with discrete-time dynamical systems (diffeomorphisms) whereas in the review we have mostly talked about continuous-time dynamical systems (flows). The latter is more in keeping with the origins of the subject while the former is more transparent technically. Suffice it to say that all theory in the book has an analog in the continuous-time case but the definitions and theorems are not quite so neat.

The book begins with a chapter on topological dynamics which defines various kinds of recurrence. Subsequent chapters discuss hyperbolicity of invariant sets and its consequences. There are careful proofs that the unstable manifolds described above are indeed manifolds (the book works with stable manifolds, but a stable manifold of f is an unstable manifold of f^{-1}), describes in detail the spectral decomposition, and proves some of the stability results mentioned above. In addition it explains the *Markov partitions* of Bowen, which can be used to show that the restriction of f to one of the Ω_i is topologically conjugate to a subshift of finite type in case the Ω_i is zero-dimensional. (This can be viewed as giving a description of the local behavior of f near Ω_i.)

The selection of material is in my view excellent for a graduate course in the subject: the student can confront most of the problems in the subject honestly but without being bogged down in too many technical details. The book grew out of a set of lecture notes from the academic year 1976–77 and there is no mention of any work which occurred subsequent to that year. Each chapter contains an annotated list of references which can be used to find proofs not included in the book, but no attempt is made to fit the work into a broader historical and mathematical context.

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