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BOUNDED GEODESICS FOR THE ATIYAH-HITCHIN METRIC

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ABSTRACT. The Atiyah-Hitchin metric has bounded geodesics which describe bound states of a monopole pair.

Introduction. The dynamics of two nonrelativistic BPS monopoles was described by Manton [**1**] as the geodesic flow on the space of collective coordinates of the monopoles M_2^0 with a special metric found explicitly by Atiyah and Hitchin [**2**]. Gibbons and Manton [**3**] studied the asymptotic metric (the Taub-Nut metric) and using its additional symmetry they integrated the equations of geodesics. They found in particular quasiperiodic solutions which describe bound states of a pair of monopoles. It is thus natural to treat the Atiyah-Hitchin metric as a small perturbation of the Taub-Nut metric and to apply the KAM theory to establish the existence of quasiperiodic geodesics. In the present note we sketch an implementation of this idea. The detailed exposition will appear elsewhere.

1. Analytic description of the Atiyah-Hitchin metric on M_2^0. The Atiyah-Hitchin metric on the four-dimensional manifold M_2^0 admits SO(3) as a symmetry group and the orbits of the action are nondegenerate, i.e., 3-dimensional with only one exception. Hence we can identify the tangent space to the orbit with the Lie algebra so(3) and write the metric in the form

$$ds^2 = f^2 d^2\eta + a^2\sigma_1^2 + b^2\sigma_2^2 + c^2\sigma_3^2, \tag{1}$$

where η is a transversal coordinate, and σ_1, σ_2 and σ_3 are the standard one-forms in so(3)*.

a, b, c and f are functions of η which can be described in the following way.

$$\begin{aligned} a^2 &= 4K(K-E)(E-Kk'^2)/E, \\ b^2 &= 4K(K-E)E/(E-Kk'^2), \\ c^2 &= 4KE(E-Kk'^2)/(K-E), \end{aligned} \tag{2}$$

where

$$K(k) = \int_0^{\pi/2} (1-k^2\sin^2\phi)^{-1/2}\,d\phi, \qquad E(k) = \int_0^{\pi/2} (1-k^2\sin^2\phi)^{1/2}\,d\phi$$

are complete elliptic integrals and $k' = \sqrt{1-k^2}$ is the conjugate modulus.

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In the formulas (2) k is assumed to be a function of η which can be chosen arbitrarily for the price of changing the function f appropriately. Gibbons and Manton [3] suggested the use of $\eta = 2K(k)$, $\pi < \eta < +\infty$, which leads to

$$f^2 = (K - E)E/K(E - Kk'^2). \tag{3}$$

2. The reduced hamiltonian system. The SO(3) symmetry of the Atiyah-Hitchin metric (1) allows the reduction of the geodesic equations to the Euler type equations

$$\begin{gathered} \frac{dM_1}{dt} = \left(\frac{1}{b^2} - \frac{1}{c^2}\right) M_2 M_3, \qquad \frac{dM_2}{dt} = \left(\frac{1}{c^2} - \frac{1}{a^2}\right) M_3 M_1, \\ \frac{dM_3}{dt} = \left(\frac{1}{a^2} - \frac{1}{b^2}\right) M_1 M_2, \\ \frac{d\eta}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial \eta}, \end{gathered} \tag{4}$$

where

$$H = \frac{1}{2}\left(\frac{p^2}{f^2} + \frac{M_1^2}{a^2} + \frac{M_2^2}{b^2} + \frac{M_3^2}{c^2}\right).$$

Geometrically the reduction means that the geodesic flow factors onto the system (4). In particular the metric (1) has bounded geodesics if and only if the system (4) has solutions bounded in η.

$M^2 = M_1^2 + M_2^2 + M_3^2$ and H are first integrals of the system. Without loss of generality we can set $M^2 = 1$. Then the system can be put into an explicitly hamiltonian form by letting

$$M_1 = \sqrt{1 - M_3^2}\cos\phi, \qquad M_2 = \sqrt{1 - M_3^2}\sin\phi,$$

which leads to the system

$$\frac{d\eta}{dt} = \frac{\partial H}{\partial p}, \quad \frac{d\phi}{dt} = \frac{\partial H}{\partial M_3}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial \eta}, \quad \frac{dM_3}{dt} = -\frac{\partial H}{\partial \phi}, \tag{5}$$

where

$$H = \frac{1}{2}\left[\frac{p^2}{f^2} + \frac{(1 - M_3^2)\cos^2\phi}{a^2} + \frac{(1 - M_3^2)\sin^2\phi}{b^2} + \frac{M_3^2}{c^2}\right].$$

Transforming H further, we get $H = H_0 + H_1$, where

$$\begin{aligned} H_0 &= \frac{1}{2}\left[\frac{p^2}{f^2} + \frac{1}{2}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)(1 - M_3^2) + \frac{M_3^2}{c^2}\right], \\ H_1 &= \frac{1}{4}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)(1 - M_3^2)\cos 2\phi. \end{aligned}$$

The hamiltonian H_0 does not depend on ϕ so it defines an integrable hamiltonian system, the other integral being M_3. Moreover there are bounded quasiperiodic motions in the system. Indeed

$$H_0 = \frac{1}{2}\frac{p^2}{f^2} + V(\eta, M_3), \qquad V(\eta, M_3) = \frac{1}{4}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)(1 - M_3^2) + \frac{M_3^2}{2c^2}$$

and for a fixed value of M_3, V has a global minimum $V_0(M_3)$, at least for small values of M_3. To see this note that $1/a^2 + 1/b^2$ is decreasing to zero and $1/c^2$ is increasing to some positive value as $\eta \to +\infty$. The manifold $\{H_0 = \text{const}, M_3 = \text{const}\}$ for values of H_0 close to $V_0(M_3)$ is compact and hence by the Liouville-Arnold Theorem it must be the torus carrying a quasiperiodic motion.

3. KAM theory. We want to treat the hamiltonian system (5) as a perturbation of the integrable system with the hamiltonian H_0. To apply the KAM theory we have to find the action-angle variables for H_0 and to estimate the perturbation.

Expanding K in the conjugate modulus k' we have

$$K = -\ln(k'/4)(1 + O(k'^2)) + O(k'^2) \quad \text{as } k' \to 0.$$

Hence

$$k'^2 = O(e^{-\eta}) \quad \text{as } \eta \to +\infty. \tag{6}$$

Also

$$E = 1 + \ln(k'/4)O(k'^2) + O(k'^2) \quad \text{as } k' \to 0$$

so that

$$E = 1 + O(\eta e^{-\eta}) \quad \text{as } \eta \to +\infty. \tag{7}$$

Applying (6) and (7) to the formulas (2) and (3), we get

$$\begin{aligned} \frac{1}{4}\left(\frac{1}{a^2} + \frac{1}{b^2}\right) &= \frac{1}{2\eta(\eta-2)} + O(\eta^{-2}e^{-\eta}), \\ \frac{1}{2}\frac{1}{c^2} &= \frac{1}{8}\frac{\eta-2}{\eta} + O(\eta e^{-\eta}), \\ \frac{1}{f^2} &= \frac{\eta}{\eta-2} + O(\eta e^{-\eta}), \quad \text{and} \\ \frac{1}{a^2} - \frac{1}{b^2} &= O(\eta^{-1}e^{-\eta}). \end{aligned} \tag{8}$$

We introduce the integrable hamiltonian

$$H_{00} = \frac{1}{2}p^2\frac{\eta}{\eta-2} + \frac{1}{8}M_3^2 + \frac{1}{4}\left(\frac{1}{\eta-2}(1-M_3^2) - \frac{1}{\eta}\right).$$

By (8) $H_{01} \equiv H_0 - H_{00} = O(\eta e^{-\eta})$ and also $H_1 = O(\eta^{-1}e^{-\eta})$.

We will find the action variables I, J for the hamiltonain H_{00} in the region of the phase space where the motion is bounded. Let $H_{00} = \frac{1}{2}p^2\eta/(\eta-2) + W(\eta, M_3)$ and let $\eta_0(M_3)$ be the value of η at which W attains its minimum, i.e., $(\partial W/\partial\eta)(\eta_0, M_3) = 0$. We will use $\varepsilon = 1/\eta_0$ as a small parameter. We have

$$\varepsilon - \varepsilon^2 = \tfrac{1}{4}M_3^2.$$

We choose the basic cycles for the torus $\{M_3 = \text{const}, H_{00} = \text{const}\}$ to be $\gamma_1 = \{M_3 = \text{const}, p = \text{const}, \eta = \text{const}, 0 \le \phi \le 2\pi\}$ and $\gamma_2 = \{M_3 = \text{const},$

$\phi = \text{const}, H_{00} = \text{const}\}$ and then

$$I = \frac{1}{2\pi}\int_{\gamma_1} p\,d\eta + M_3\,d\phi = M_3,$$
$$J = \frac{1}{2\pi}\int_{\gamma_2} p\,d\eta + M_3\,d\phi = \frac{1}{2\pi}\int_{\gamma_2} p\,d\eta.$$

To evaluate the last integral we make for a fixed M_3 the change of variables $\eta = (1+\tilde{\eta})/\varepsilon$, $p = \varepsilon\tilde{p}$. We have

$$H_{00} = \frac{1}{2}\varepsilon - \varepsilon^2 + \frac{1}{2}\varepsilon^2\left[\left(\frac{\tilde{\eta}}{\tilde{\eta}+1}\right)^2 + \tilde{p}^2\right]\left(1 + \frac{2\varepsilon}{\tilde{\eta}+1-2\varepsilon}\right)$$

and $J = (1/2\pi)\int_{\gamma_2}\tilde{p}\,d\tilde{\eta}$.

By straightforward integration

$$J = (1-c)^{-1/2} - 1 + O(\varepsilon),$$

where

$$c = \left[\left(\frac{\tilde{\eta}}{\tilde{\eta}+1}\right)^2 + \tilde{p}^2\right]\left(1 + \frac{2\varepsilon}{\tilde{\eta}+1-2\varepsilon}\right).$$

Finally

$$H_{00} = \frac{1}{2}(\varepsilon - \varepsilon^2) - \frac{1}{2}\varepsilon^2\frac{1}{(J+1)^2} + O(\varepsilon^3).$$

Switching back to $M_3 = I$, we get

$$H_{00} = \frac{1}{8}I^2 - \frac{1}{32}\frac{I^4}{(J+1)^2} + O(I^6).$$

For small I where the perturbation $H_{01} + H_1$ is small the hamiltonian H_{00} is degenerate in the sense that the hessian $\det(\partial^2 H/\partial(I,J)^2)$ is also small, so that the standard KAM theory does not work in our case. The appropriate version of the KAM theory was actually developed by Arnold [4] and we can conclude that for sufficiently small I_0, for most initial conditions in the domain $0 < I \le I_0$ (in the sense of Lebesgue measure) the motion is quasiperiodic and close for all $-\infty < t < +\infty$ to the motion

$$\dot{I} = 0, \quad \dot{J} = 0, \quad \dot{\phi}_I = \frac{\partial H_{00}}{\partial I}, \qquad \dot{\phi}_J = \frac{\partial H_{00}}{\partial J}$$

with appropriate initial conditions. ϕ_I and ϕ_J are the angle variables conjugate to I and J respectively.

Conclusion. The Atiyah-Hitchin metric has many bounded geodesics. Their union in the phase space has positive Lebesgue measure.

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