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ONE-DIMENSIONAL DYNAMICS: THE SCHWARZIAN DERIVATIVE AND BEYOND

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Most of the important results in the study of the dynamics of smooth interval maps $f\colon [0,1] \to [0,1]$ assume the condition that $Sf < 0$ where Sf is the Schwarzian derivative of f:

$$Sf = \frac{D^3 f}{Df} - \frac{3}{2}\left(\frac{D^2 f}{Df}\right)^2 .$$

This condition, although very powerful, has the disadvantage of being too restrictive and, even worse, it is not invariant under C^∞ change of coordinates. More precisely, there exists a C^∞ diffeomorphism $\varphi\colon [0,1] \to [0,1]$ such that $\varphi f \varphi^{-1}$ does not have negative Schwarzian derivative.

In this announcement we will present a technique which enables one to replace these conditions by smoothness conditions: we assume that f is C^3 and that f is nonflat at the critical points (i.e. f is C^∞ near the critical points and at each critical point one of the derivatives is nonzero). We will illustrate this technique by showing the analogue, for maps $f\colon [0,1] \to [0,1]$ with one critical point, of the result of Denjoy done for C^2 circle-diffeomorphisms.

More precisely, Denjoy showed that a C^2 diffeomorphism $f\colon S^1 \to S^1$ cannot have any wandering interval $L \subset S^1$. Here, we say that L is a *wandering interval* if L, $f(L)$, $f^2(L), \ldots$ are mutually disjoint and no point $x \in L$ is asymptotic to a periodic orbit. From this it follows that if f is a C^2 diffeomorphism, then either f has a periodic orbit or it is conjugate to a rigid rotation. We say that $f\colon [0,1] \to [0,1]$ is in class $\mathcal{A}$ if f is a C^3 map with only one critical point and f is nonflat at its critical point.

THEOREM. *Let $f\colon [0,1] \to [0,1]$ be in class $\mathcal{A}$. Then f has no wandering intervals.*

COROLLARY. *Every f in $\mathcal{A}$ is semiconjugate to a map from the quadratic family $f_\lambda\colon [0,1] \to [0,1]$ defined by $f_\lambda(x) = \lambda x(1-x)$. This semiconjugacy only collapses the basin of attraction of the periodic orbits which do not attract the critical point.*

REMARK 1. The Schwarzian derivative was introduced in one-dimensional dynamics by D. Singer [**S**]. Guckenheimer proved the nonexistence of wandering intervals for maps in $\mathcal{A}$ under the assumption that $Sf < 0$ [**G**].

REMARK 2. In general a map f in $\mathcal{A}$ can have several attracting periodic orbits, whereas if f_λ has an attracting periodic point then it attracts the

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critical point $\frac{1}{2}$. It follows that one cannot hope to get a conjugacy between f and f_λ.

REMARK 3. We expect to be able to prove that there is a bound for the period of the attracting periodic orbits of each map f in $\mathcal{A}$. This would imply that the semiconjugacy only collapses a finite number of intervals and their backward orbits.

SKETCH OF THE PROOF. We consider two cross-ratios. Let $J, T \subset [0,1]$ be open intervals such that $\mathrm{Clos}(T) - J$ has two connected components L and R. We define

$$C(T,J) = \frac{|J|\,|T|}{|L \cup J|\,|J \cup R|} \quad \text{and} \quad D(T,J) = \frac{|J|\,|T|}{|L|\,|R|},$$

where $|J|$ denotes the length of the interval J. If $g\colon [0,1] \to [0,1]$ is monotone on T we define the operators

$$A(g,T,J) = \frac{C(g(T), g(J))}{C(T,J)} \quad \text{and} \quad B(g,T,J) = \frac{D(g(T), g(J)}{D(T,J)}.$$

If g has negative Schwarzian derivative we can see that $A(g,T,J) > 1$ and $B(g,T,J) > 1$. In the general case we prove the following:

THEOREM 1. *Let $f\colon [0,1] \to [0,1]$ be a C^∞ map whose critical points are nonflat. There exist $\delta > 0$ and $\frac{1}{18} > \epsilon > 0$ such that if $T \supset J$ are open intervals satisfying*: (i) *f^m is a diffeomorphism on* $\mathrm{Clos}(T)$; (ii) $\sum_{k=0}^{\infty} |f^k(J)| < \delta$; (iii) $|L|\,|R| < \epsilon |J|^2$ *then*

$$A(f^m, T, J) < 1 - \frac{8|L|\,|R|}{|J|^2}.$$

COROLLARY. *Under the conditions of Theorem* 1 *we have*

$$\frac{|f^m(L)|\,|f^m(R)|}{|L||R|} < \frac{18}{|J|^2} |f^m(J)|\,|f^m(T)|.$$

THEOREM 2. *Let $f\colon [0,1] \to [0,1]$ be a C^∞ map whose critical points are nonflat. There exists a constant $C_1 > 0$ such that if $T \supset J$ are intervals such that* (i) *f^m is a diffeomorphism on* $\mathrm{Clos}(T)$; (ii) $\sum_{k=0}^{m} |f^m(T)|^2 = S < 3$, *then*

$$\operatorname{Log} B(f^m, T, J) > -C_1 S.$$

THEOREM 3. *Let $f\colon [0,1] \to [0,1]$ be a C^∞ map whose critical points are nonflat. Let C_1 be as in Theorem* 2. *If $T = [a,b] \subset [0,1]$ is such that f^m is a diffeomorphism on T and $\sum_{i=0}^{m-1} |f^i(T)| = \delta < 1$ then*

$$|Df^m(x)| \geq (\mathrm{Exp}(-C_1 S))^3 |Df^m(a)|$$

or

$$|Df^m(x)| \geq (\mathrm{Exp}(-C_1 S))^3 |Df^m(b)|$$

or both.

Suppose, by contradiction, that f has a wandering interval J. By replacing J by some iterate we may assume that $\sum_{k=0}^{\infty} |f^k(J)| < \delta$ and $f^n(\mathrm{Clos}(J))$ does not contain the critical point c for every n. By the theorem of Schwartz

[**CE**, pp. 111], the forward iterates of J must accumulate at the critical point c. Hence we may define a sequence of integers $k(n)$ by $k(0) = 0$ and $k(n) = \min\{k; f^{k(n-1)}(J) \supset \langle f^{k(n-1)}(J), (f^{k(n-1)}(J))'\rangle\}$. Here, for an interval T which does not contain the critical point, T' denotes the interval $f^{-1}(f(T))-T$ and $\langle T, T'\rangle$ is the smallest interval containing $T \cup T'$. Let $V_n = \{x; f^n(x) \in \operatorname{int}(\langle x, x'\rangle)$ and $f^i(x) \notin \langle x, x'\rangle$ for $i < n\}$. As in [**G**], the image of the boundary points of each connected component of V_n are fixed points of f^n. Furthermore, the first $n-1$ iterates of such a connected component are disjoint intervals. Using these facts for the connected component of $V_{k(n+1)-k(n)}$ containing $f^{k(n)}(J)$, and Theorem 3, we get that there is a constant $e > 0$ independent of n such that

$$|f^{k(n+1)}(J)| > e|f^{k(n)}(J)|.$$

Let K_n be the largest interval containing J on which f^n is monotone. Since J is a wandering interval we have that $K_n - J = L_n \cup R_n$, where R_n and L_n are nonempty intervals whose lengths go to zero as n goes to infinity. As in [**G**], we get that $f^{k(n)}(K_{k(n)})$ contains either $f^{k(n-1)}(J)$ or $(f^{k(n-1)}(J))'$ and it contains also either $f^{k(n+1)}(J)$ or $(f^{k(n+1)}(J))'$. Hence, by interchanging $L_{k(n)}$ with $R_{k(n)}$ if necessary, we get

$$|f^{k(n)}(L_{k(n)})| > \alpha|f^{k(n-1)}(J)|$$

and

$$|f^{k(n)}(R_{k(n)})| > \alpha|f^{k(n+1)}(J)| > e\alpha|f^{k(n)}(J)|,$$

where $\alpha = \inf|Df(x)|/|Df(x')|$. Since $|f^{k(n)}(J)| \to 0$ as $n \to \infty$ we may choose a subsequence $n(i) \to \infty$ such that $|f^{k(n(i))}(J)| > |f^{k(n(i-1))}(J)|$. From the corollary of Theorem 1 we get

$$\frac{|f^{k(n(i))}(L_{k(n(i))})|\,|f^{k(n(i))}(R_{k(n(i))})|}{|L_{k(n(i))}|\,|R_{k(n(i))}|} \leq \frac{18}{|J|^2}|f^{k(n(i))}(J)|\,\{(|f^{k(n(i))}(L_{k(n(i))})| + |f^{k(n(i))}(J)| + |f^{k(n(i))}(R_{k(n(i))})|\}.$$

By shrinking $K_{n(i)}$ we get that

$$\frac{|f^{k(n(i))}(L_i^*)|\,|f^{k(n(i))}(R_i^*)|}{|L_{k(n(i))}|\,|R_{k(n(i))}|} \leq \frac{|f^{k(n(i))}(L_i^*)|\,|f^{k(n(i))}(R_i^*)|}{|L_i^*|\,|R_i^*|} \leq \frac{18}{|J|^2}|f^{k(n(i))}(J)|\,\{|f^{k(n(i))}(L_i^*)| + |f^{k(n(i))}(J)| + |f^{k(n(i))}(R_i^*)|\}$$

for every $K_i^* = L_i^* \cup J \cup R_i^* \subset K_{n(i)}$. Choose L_i^* and R_i^* so that

$$|f^{k(n(i))}(L_i^*)| = \min\{|f^{k(n(i))}(J)|, \alpha|f^{k(n(i-1))}(J)|\}$$

and

$$|f^{k(n(i))}(R_i^*)| = \min\{|f^{k(n(i))}(J)|, e\alpha|f^{k(n(i))}(J)|\}.$$

Then

$$\frac{|f^{k(n(i))}(L_i^*)|\,|f^{k(n(i))}(R_i^*)|}{|L_{k(n(i))}|\,|R_{k(n(i))}|} \leq 3\frac{18}{|J|^2}|f^{k(n(i))}(J)|^2,$$

$$\frac{|f^{k(n(i))}(J)|}{|f^{k(n(i))}(R_i^*)|} = \max(1,(e\alpha)^{-1})$$

and

$$\frac{|f^{k(n(i))}(J)|}{|f^{k(n(i))}(L_i^*)|} \leq \max(1,\alpha^{-1})$$

because $|f^{k(n(i-1))}(J)| > |f^{k(n(i))}(J)|$. Hence

$$\frac{1}{|L_{k(n(i))}|\,|R_{k(n(i))}|} \leq 3\frac{18}{|J|^2}\max(1,(e\alpha)^{-1})\max(1,\alpha^{-1}).$$

This is a contradiction because $|L_{k(n(i))}|$ and $|R_{k(n(i))}|$ go to zero as $n(i) \to \infty$.

REFERENCES

[C-E] P. Collet and J. Eckmann, *Iterated maps of the interval as dynamical systems*, Birkhäuser, 1980.

[D] A. Denjoy, *Sur les courbes définié par les équations différentielles à la surface du tore*, J. Math. Pures Appl. **11** (1932), ser. 9.

[G] J. Guckenheimer, *Sensitive dependence on initial conditions for one-dimensional maps*, Comm. Math. Phys. **70** (1979), 133–160.

[Si] D. Singer, *Stable orbits and bifurcations of maps of the interval*, SIAM J. Appl. Math. **35** (1978), 260.

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