BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 18, Number 1, January 1988
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0273-0979/88 $1.00 + $.25 per page

Geometric theory of foliations, by César Camacho and Alcides Lins Neto. Translated from Portuguese by Sue E. Goodman. Birkhäuser, Boston, Basel and Stuttgart, 1985, 205 pp., $42.00. ISBN 0-8176-3139-9

The word "foliation" stands out in the title of the book under review as requiring explanation. For those who are predisposed to botanical terminology (this definitely includes the reviewer) this term conjures up images which are perhaps reminiscent of Desargues [**20**]. The term "structure feuilletée" was coined by Georges Reeb [**2**, **16**]. Later authors began using the briefer "feuilletage" which was translated as "foliation." Reeb winces at the botanical interpretation and offers instead a gastronomic motivation [**18**]:

> "A défaut de botaniste l'assemblée comptera peut-être un pâtissier. La pâte feuilletée—j'ai de bonnes raisons de le croire—donne une bonne idée d'un feuilletage (de codimension 1 dans R^3) dont elle dessine bien les feuilles et en suggère des propriétés."

Of course the more mundane translation to "sheeted" or "layered" shows that the terminology is appropriate since it suggests the image of pages in a book. Unlike the ill-fated terminology of Desargues, "foliation" has become a mathematical household word.

A foliation is simply a decomposition of a manifold into a disjoint union of immersed submanifolds (called leaves) of constant dimension such that the decomposition is locally homeomorphic to the decomposition of $R^n = R^k \times R^{n-k}$ into the parallel submanifolds $R^k \times \{\text{point}\}$. The most classical version of this is the "flow box" neighborhood of a point at which a vector field is nonzero ($k = 1$). With this case in mind the study of foliations may be thought of as a generalization of Poincaré's study of differential equations from the dynamic viewpoint [**14**]. One of the charming aspects of this subject is that there are several sources of historical motivation. Reeb, for example, derives inspiration from the study of differential equations in the complex domain inaugurated by Painlevé. Differential equations is not the only field which can claim the study of foliations. Geometers and topologists can also participate and claim antecedents such as É. Cartan, C. Ehresmann, H. Hopf, and H. Kneser.

The book under review deals with the principal results obtained in the theory of foliations during the period 1947–1965. The earliest of these results were the discovery of the Reeb foliation and the Reeb stability theorems [**17**]. H. Hopf had asked whether there is a completely integrable plane field on the 3-sphere. Reeb answered this question by constructing the now well-known foliation having a single toral leaf with all other leaves planar and spiralling towards the toral leaf. This example provided the justification for the further qualitative study of foliations. Furthermore, its significance in motivating later work of Haefliger and Novikov cannot be overstated.

The Reeb stability theorems give information about leaves near a compact leaf. The local version, which is true for any codimension, says that some neighborhood of a compact leaf having finite fundamental group consists entirely of compact leaves having finite fundamental groups. The arguments used in the proof are variants of those associated with the theory of covering spaces. For foliations of codimension one of a compact manifold the neighborhood can be taken to be the entire manifold; that is, every leaf of the foliation is compact. Reeb also showed that similar statements hold for perturbations of the original foliation. In [**3**] Ehresmann and Shih observed that the hypothesis of having a finite fundamental group for the leaf can be weakened. This involved the notion of holonomy. Based loops in the original leaf can be lifted to nearby leaves in an obvious way which turns out to be independent of the homotopy class of the loop. This procedure is a generalization of the classical Poincaré map of a periodic orbit of a flow, where the periodic orbit is the loop and the lifted path is a nearby (generally not closed) trajectory of approximately the same length. In this way, the fundamental group of a leaf determines, by homomorphic image, a group of germs of local homeomorphisms of R^{n-k} which fix the origin. This image group (actually its isomorphism class) is called the holonomy group of the leaf. The term holonomy is from differential geometry and goes back to É. Cartan. Ehresmann's motivation in selecting this term is the analogy between a foliation and an integrable (i.e., flat) connection. With these refinements the local Reeb stability theorem takes the following form.

THEOREM. *If L is a compact leaf of a foliation which has a finite holonomy group then there is a neighborhood of L consisting entirely of compact leaves having finite holonomy groups. Furthermore, in a sufficiently small neighborhood of L the compact leaves will all be covering spaces of L.*

The thesis of Haefliger [**7**] contains further important advances in the qualitative study of foliations. One of these was the first essential use of the classical arguments of Poincaré and Bendixson to prove the following result concerning real analytic foliations of codimension one.

THEOREM (HAEFLIGER). *If a compact manifold admits a real analytic foliation of codimension one then the manifold has infinite fundamental group.*

In particular, simply connected manifolds such as the 3-sphere do not have analytic foliations of codimension one. It was already clear that the Reeb foliation of the 3-sphere is not analytic because the holonomy group of the toral leaf contains germs which are nontrivial even though they are the identity on an open subinterval. It is precisely this phenomenon which Haefliger's proof detects. If a simple closed curve is everywhere transverse to the foliation then it bounds an immersed disk which can be perturbed to be in general position with respect to the foliation. This means that the singular foliation induced on the disk has only Morse singularities (centers and saddles). This idea of using surfaces in general position with respect to a codimension one foliation was a significant advance of the theory.

Another major advance was made by Novikov [**12**]. Novikov's paper stands out from the rest of the literature in this field as having an unusually high ratio of inspiration to perspiration. The most famous results in this paper are the compact leaf theorems. Kneser [**11**] had shown that every foliation of the Klein bottle having one-dimensional leaves must have a compact leaf (circle). Novikov showed that a codimension one foliation of a compact 3-manifold having finite fundamental group must have a compact leaf; in fact, such a foliated manifold must contain a solid torus foliated in the same manner as the Reeb foliation. The techniques used by Novikov were similar to those of Haefliger except that the phenomenon being sought was different. Novikov showed that the hypotheses imply the existence of a one-parameter family of embedded circles C_t ($t \geq 0$), each of which is contained in a single leaf, such that for $t > 0$, C_t is homotopic to zero in its leaf whereas C_0 is not. In the subsequent literature C_0 is usually called a vanishing cycle. Novikov also showed that any codimension one foliation of a compact 3-manifold having nonzero second homotopy group must have a compact leaf (but not necessarily a torus). The arguments in this case start with an embedded 2-sphere in general position with respect to the foliation. The compact leaf theorems take the following amalgamated form.

THEOREM (NOVIKOV). *If M is a compact 3-manifold whose universal covering space is not contractible then every codimension one foliation of M has a compact leaf.*

Significant advances in understanding the dynamics of codimension one foliations were made by Sacksteder [**15**]. A minimal set of a foliation is a nonempty compact set which is a union of leaves and which has no proper subset satisfying these conditions. This notion was originally introduced by G. D. Birkhoff in the context of flows. Understanding the nature of minimal sets is the first step toward describing the structure of a foliation. In [**1**] A. Denjoy had shown that a flow on the 2-torus which is twice continuously differentiable and without stationary points can only have either a single periodic orbit or the entire torus as a minimal set. Denjoy also gave a counterexample for the C^1 case, that is, a flow containing a unique minimal set which is homeomorphic locally to the product of a Cantor set and an interval. Inspired by Denjoy's work and its generalization to flows on arbitrary surfaces by A. J. Schwartz, Sacksteder obtained the following fundamental result.

THEOREM (SACKSTEDER). *Suppose that M is a minimal set of a codimension one foliation of class C^2 which is neither a single compact leaf nor the entire manifold. Then for some leaf in M there is an element in its holonomy group whose derivative (at the fixed point corresponding to the leaf) has absolute value < 1. In particular, the fixed point is contracting and isolated and the leaf has nontrivial holonomy and fundamental groups.*

In the classical case the only possibilities for the leaf are the line, which is simply connected, and the circle, so Sacksteder's result is a generalization of Denjoy's.

Activity in the study of foliations has increased dramatically since 1965. Evidence of this is provided by the massive bibliography compiled by Godbillon [**5**]. In the process new directions of inquiry have been pursued. Among these are the study of characteristic classes and classifying spaces for foliations, questions of existence (on a particular manifold or even in a particular homotopy class of k-plane fields), and geometric properties of an asymptotic nature (e.g., curvature and growth properties of leaves). However, these topics are beyond the scope of the book under review and will not be discussed here.

The present book provides a detailed introduction to the topics described above. It also includes introductory material on differentiable manifolds and an appendix on the Frobenius theorem. The book is divided into eight chapters, four of which have extensive notes at the end. These notes describe additional results; some are in survey format. The book concludes with a list of exercises. The original version of the book, in Portuguese, appeared in the series Projeto Euclides written for study by students in Brazil. The English translation, by S. Goodman, is excellent. On the dust jacket the book takes credit for being "a valuable tool for students and those studying the theory of foliations, as well as a comprehensive reference source for research libraries." The first of these assertions, which is compatible with the authors' intentions as set forth in the introduction, is definitely valid. There are very detailed proofs of the results of Reeb, Haefliger, and Novikov described above. Although Sacksteder's theorem is not proved, the chapter notes contain a statement of the result as well as a detailed presentation of the Denjoy counterexample. In addition, Chapter 5 contains an example, due to Sacksteder, of a C^∞ foliation of codimension one which possesses a nowhere dense minimal set which is not a single compact leaf. The style of exposition is leisurely and illuminating and brought back to this reviewer pleasant memories of his first course in differential topology (taught by E. Lima). The authors' particular interest in singularities of differential forms is shown in some of the chapter notes which contain material one is not likely to see in other books on foliations.

The publisher's assertion that this book is a comprehensive reference source is unjustified hyperbole. Although a good introduction, the book was not really up to date when written and, in some instances, the exposition lacks polish. For example, the authors seem to convey the impression that if the union of two foliation charts is contained in a third foliation chart then any plaque of the first chart intersects at most one plaque of the second. That this is false is easily seen from examples of codimension two. This causes the argument at the bottom of p. 62 to be flawed. Later on the authors offer a different argument requiring differentiability which seems adequate. In general the use of differentiability in proofs is pedagogically sound, especially when the continuous case is much more work. In Chapters 6 and 7, however, the authors assume the foliations to be of class C^2 in the use of general position arguments for surfaces. The C^1 case would have not have been much harder using the arguments of [**4**]. The chapter on actions by Lie groups (Chapter 8) is rather half-baked. Several proofs are given of Lima's result that the 3-sphere cannot have two commuting vector fields which are everywhere linearly independent. This is an important result, but it was eclipsed by the work of Novikov. The most important result in this chapter, beyond Lima's theorem, is the observation that a locally free action of codimension one cannot have vanishing cycles, but no reference is given for this result. A suitable reference would have been [**13**] (where the vanishing cycle result is attributed to Roussarie) which also contains a more general extension of Lima's result than that presented in this book.

The book would have been more up to date had the chapter notes continued past Chapter 5. The book does not mention the compact leaf theorem of Kneser [**11**] even though this reference is cited in another context. Most astonishing of all is that the authors never mention the counterexamples to compact leaf theorems in higher codimension discovered by their neighbor P. Schweitzer [**19**]! Also, the bibliography at the end of the book was neither translated nor updated. For example, we are still offered a Spanish translation of Seifert and Threfall (sic), and all works cited after 1976 are of Brazilian origin.

There are other sources with which the present book should be compared. The expository papers by Haefliger [**8**, **9**] still provide an excellent crash-course, but the book under review is much more detailed. Readers wanting a more comprehensive treatment, which takes the subject past 1970, should study the multiple-volume works of Godbillon [**6**] and Hector and Hirsch [**10**]. The reader who still hasn't had enough can read all of the papers listed in Godbillon's bibliography [**5**].

To summarize, the Camacho-Neto book is a detailed but outdated introduction to the geometric (qualitative) study of foliations. Given the price of the translated version, the enterprising student of foliations might consider reading the original version. This would allow her or him to learn some mathematical Portuguese along the way. This is a useful endeavor, since many, if not most, of those who work in this field have paid mathematical visits to either I.M.P.A. or P.U.C. in Rio de Janeiro.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 18, Number 1, January 1988
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0273-0979/88 $1.00 + $.25 per page

Symplectic techniques in physics, by Victor Guillemin and Shlomo Sternberg, Cambridge University Press, Cambridge, London, New York, New Rochelle, Melbourne, Sydney, 1986, xi + 468 pp., $49.50. ISBN 0-521-24866-3

This book, aimed at a mixed public of physicists and mathematicians, starts with a 150-page chapter, called the Introduction, on optics. The purpose of this introduction is to illustrate the importance of symplectic geometry for physics.

The primitive way in which the discussion starts with Gaussian optics may serve as an eye-opener for some. However, it may also make a somewhat artificial impression on those who already have seen the relation between Snell's law, Fermat's principle, and Hamilton's treatment of geometric optics in some early physics course.

The subsequent epic story of Fresnel's discovery of the wave nature of light, leading to oscillatory integrals, is always very impressive. Here I am curious whether Fresnel himself really used complex notation as suggested in the book. Also I missed the end of the story, explaining light, and in particular its polarization, as rapidly oscillating solutions of Maxwell's equations.

Instead the book takes a surprising and exciting turn to use Fresnel integrals in order to pass to the standard representation of the metaplectic group on $L^2(\mathbf{R}^n)$. This move into quantum mechanics is followed by, among other things, a discussion of the Groenwald–van Hove theorem, saying that there is no way of extending the metaplectic representation to include any non-quadratic polynomial. It is a great service to the public to treat this subject so completely in this book.

Maxwell's equations do appear at the end of Chapter I, but only to treat the motion of a charged particle in an electromagnetic field as being Hamiltonian with respect to a symplectic form on the cotangent bundle, which differs from the standard one by a "magnetic term".

Chapter I is concluded with the provoking question "Why symplectic geometry?" I would like to add to their answer that classical mechanics not