11. V. I. Arnol'd, *Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits*, Ann. Inst. Fourier Grenoble **16** (1966), 319–361.

12. ______, *The Hamiltonian nature of the Euler equations in the dynamics of a rigid body and an ideal fluid*, Uspekhi Mat. Nauk **24** (1969), 225–226 (Russian).

13. C. S. Gardner, *Korteweg-de Vries equation and generalizations*. IV, *The Korteweg-de Vries equation as a Hamiltonian system*, J. Math. Phys. **12** (1971), 1548–1551.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 18, Number 1, January 1988

0273-0979/88 $1.00 + $.25 per page

The geometry of discrete groups, by Alan F. Beardon. Graduate Texts in Mathematics, vol. 91, Springer-Verlag, Berlin and New York, 1983, xii + 337 pp., $39.00. ISBN 0-387-90788-2

Let $\overline{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$ be the one-point compactification of $\mathbf{R}^n$, $n \geq 1$. The group G_n of Möbius transformations is the transformation group on $\overline{\mathbf{R}}^n$ generated by the translations

$$(1) \qquad x \mapsto x + a, \qquad a \in \mathbf{R}^n,$$

and the inversion

$$(2) \qquad x \mapsto x/|x|^2$$

in the unit sphere. There are a number of reasons why Möbius transformations play a central role in the geometry of $\mathbf{R}^n$. For instance:

(a) According to a classical theorem of Liouville, if $n \geq 3$, every conformal map from one subregion of $\mathbf{R}^n$ to another is the restriction of a Möbius transformation.

(b) The sense-preserving transformations in G_2 are the fractional linear transformations

$$(3) \qquad g(z) = (az + b)(cz + d)^{-1}, \qquad ad - bc = 1,$$

which are fundamental tools in geometric function theory.

(c) If we embed $\mathbf{R}^n$ in $\mathbf{R}^{n+1}$ in the usual way, by identifying $\mathbf{R}^n$ with $(e_{n+1})^{\perp}$, formulas (1) and (2) define an action of G_n on $\overline{\mathbf{R}}^{n+1}$. In fact G_n is the subgroup of G_{n+1} that maps the half-space

$$H^{n+1} = \{x \in \mathbf{R}^{n+1}; x \cdot e_{n+1} > 0\}$$

onto itself. H^{n+1} with the Poincaré metric $ds = |dx|/(x \cdot e_{n+1})$ is the $(n+1)$-dimensional hyperbolic space, and G_n is its isometry group.

(d) Every Riemannian manifold of constant negative curvature (-1) can be represented as the quotient of H^{n+1} by a discrete subgroup Γ of G_n. In particular, the classical uniformization theorem implies that almost all

Riemann surfaces are quotients H^2/Γ, $\Gamma \subset G_1$, and fundamental work of Thurston implies that an important class of 3-manifolds have the form H^3/Γ, $\Gamma \subset G_2$. Thus the hyperbolic plane and hyperbolic 3-space have fundamental importance in low-dimensional geometry and topology.

(e) Certain subgroups Γ of G_2 act properly discontinuously on nonempty open sets D in the Riemann sphere $\overline{\mathbf{R}}^2$ so that each connected component of the quotient space D/Γ is a Riemann surface. These so-called Kleinian groups have been very useful in the theory of deformations of Riemann surfaces.

For all these reasons there has been a need for an introductory book that would make recent developments in the theory of discrete subgroups of G_n accessible to the general mathematician. Beardon's book addresses this need. After a lucid discussion of the geometry of Möbius transformations and their actions on $\overline{\mathbf{R}}^n$ and H^{n+1}, the author specializes to the low-dimensional case, studying the sense-preserving subgroups of G_2 and G_1. These consist of the fractional linear transformations (3), where the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

belongs to $\mathrm{SL}(2, \mathbf{C})$ if $g \in G_2$ and to $\mathrm{SL}(2, \mathbf{R})$ if $g \in G_1$ $(\subset G_2)$.

A fundamental property of the discrete subgroups of G_2, discovered in a more general setting by Kazhdan and Margulis (see [**8**]), is their uniform discreteness. For subgroups of G_2, this has been formulated quantitatively by Jørgensen's inequality. Call a subgroup of G_2 elementary if it leaves a one- or two-point subset of $\overline{\mathbf{R}}^3$ invariant. Suppose the matrices A and B in $\mathrm{SL}(2, \mathbf{C})$ generate a discrete nonelementary subgroup of G_2. Jørgensen's inequality states that

$$|\mathrm{tr}(A)^2 - 4| + |\mathrm{tr}(ABA^{-1}B^{-1}) - 2| \geq 1. \tag{4}$$

(Here $\mathrm{tr}(A)$ is the trace of the matrix A.) That inequality contains a great deal of information. For instance, Jørgensen used it to prove the following remarkable

THEOREM (JØRGENSEN [4]). *Let G be a nonelementary subgroup of G_2. Then G is discrete if and only if every two-generator subgroup of G is discrete. If in addition $G \subset G_1$, then G is discrete if and only if every cyclic subgroup is discrete.*

Another aspect of uniform discreteness, important in the deformation theory of Riemann surfaces, concerns short closed Poincaré geodesics on a Riemann surface X. There is a number $\delta > 0$ such that if the universal cover of X is the hyperbolic plane H^2, then every closed Poincaré geodesic on X of length less than δ is (a multiple of) a simple closed geodesic. Moreover, distinct simple closed geodesics of length less than δ are uniformly far apart. The author gives sharp geometric versions and proofs of these and related results. His treatment relies on a very detailed and useful discussion of plane hyperbolic geometry and trigonometry.

This is the first book to contain a thorough discussion of the material above, and all people who study discrete subgroups of $\mathrm{SL}(2, \mathbf{C})$ and $\mathrm{SL}(2, \mathbf{R})$ or the geometry of Riemann surfaces will find it an indispensable reference. The general reader will find the presentation lucid but will have to look elsewhere for the many applications of this material. Some of them can be found in references [**1**, **2**, **3**, **5**, **6**, and **7**] below.

REFERENCES

1. W. Abikoff, *The real analytic theory of Teichmüller space*, Lecture Notes in Math., vol. 820, Springer-Verlag, Berlin and New York, 1980.

2. A. Fathi, F. Laudenbach, V. Poenaru et al., *Travaux de Thurston sur les surfaces*, Astérisque **66–67** (1979).

3. W. J. Harvey (ed.), *Discrete groups and automorphic functions*, Academic Press, New York, 1977.

4. T. Jørgensen, *On discrete groups of Möbius transformations*, Amer. J. Math. **98** (1976), 739–749.

5. O. Lehto, *Univalent functions and Teichmüller spaces*, Springer-Verlag, 1987.

6. J. W. Morgan and H. Bass (eds.), *The Smith conjecture*, Academic Press, New York, 1984.

7. W. Thurston, *The geometry and topology of 3-manifolds*, Lecture Notes, Princeton Univ., 1980.

8. H.-C. Wang, *Discrete nilpotent subgroups of Lie groups*, J. Differential Geom. **3** (1969), 481–492.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 18, Number 1, January 1988

0273-0979/88 $1.00 + $.25 per page

Limit theorems for sums of exchangeable random variables, by Robert L. Taylor, Peter Z. Daffer, and Ronald F. Patterson. Rowman and Allanheld, Totowa, New Jersey, 1985, 152 pp., $22.50. ISBN 0-8476-7435-5

Independent, identically distributed (i.i.d.) random variables form a cornerstone of theoretical statistics and the behavior of sums of such random variables constitutes a significant portion of probability theory. A natural generalization is from i.i.d. to exchangeable random variables, that is, to random variables $\{X_n, n \geq 1\}$ whose joint distributions are invariant under finite permutations. Indeed, the two notions are intertwined in a number of ways.

If n points are selected at random in the interval $[0,1]$, then these n i.i.d. random variables (when ordered) partition the unit interval into $n+1$ subintervals whose lengths X_i, $i = 1, \ldots, n+1$ are exchangeable random variables.

Alternatively, if N balls are cast at random into n cells labelled $1, 2, \ldots, n$ and Y_j is the number of the cell containing the jth ball, then $Y_1, \ldots, Y_N$ are i.i.d. However, if X_{ni} equals 1 or 0 according as the ith cell is or is not empty, then $\{X_{ni},\ 1 \leq i \leq n, n \geq 1\}$ is a double array which, for each n, encompasses a finite collection of exchangeable random variables. The row sum $\sum_{i=1}^{n} X_{ni}$ is, of course, the number of empty cells.

If two infinite sequences of i.i.d. random variables with common distributions F and G are selected with probabilities α and $1-\alpha$ respectively then the