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ASYMPTOTICS OF SMALL EIGENVALUES OF RIEMANN SURFACES

MARC BURGER

Recently there has been a great deal of interest in geometric bounds on small eigenvalues of the Laplace operator on a Riemann surface [**S.W.Y**, **D.P.R.S.**]. Here we determine the precise asymptotic behaviour of these small eigenvalues. Let S_δ be a compact Riemann surface of genus $g \geq 2$ whose first k nonzero eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$ are small, i.e., $\lambda_k \leq \delta$ and $\lambda_{k+1} \geq c_1$. Then by [**S.W.Y.**] there exists a constant $a = a(g) > 0$ such that the closed geodesics $\gamma_1 \cdots \gamma_r$ of length less than $a \cdot \delta$ separate S_δ into $k+1$ pieces $S_1, \ldots, S_{k+1}$ and all other closed geodesics of S_δ have length greater than $a(g)$. Let Λ be the graph whose vertices are the pieces S_i. Suppose vertex S_i has mass $v_i = \mathrm{vol}(S_i)$ and the length L_{ij} of an edge joining S_i to S_j is the total length of the geodesics contained in $S_i \cap S_j$. Furthermore, let $0 < \lambda_1(\Lambda) \leq \cdots \leq \lambda_k(\Lambda)$ be the spectrum of the quadratic form $\sum (F(S_i) - F(S_j))^2 L_{ij}$ with respect to the norm $\sum F(S_i)^2 v_i$. Then one has

THEOREM 1.

$$\lim_{\delta \to 0} \frac{\lambda_j(S_\delta)}{\lambda_j(\Lambda)} = \frac{1}{\pi} \quad \textit{for all } 1 \leq j \leq k.$$

This convergence is uniform for all surfaces S_δ with $\lambda_{k+1}(S_\delta) \geq c_1$ and fixed genus.

REMARK. The fact that $\limsup_{\delta \to 0} \lambda_j(S_\delta)/\lambda_j(\Lambda) \leq 1/\pi$ follows easily from [**C.CdV**]. This paper also shows the convergence of this ratio in the case that the lengths $l(\gamma_i)$ all have the same behaviour near zero, i.e. $l(\gamma_i) = d_i \varepsilon$ for $\varepsilon \to 0$, and fixed d_i.

SKETCH OF PROOF. Complete $\gamma_1 \cdots \gamma_r$ to a set of geodesics $\gamma_1 \cdots \gamma_{3g-3}$, giving a decomposition of S into Y-pieces with length $l(\gamma_i) \leq L_g$, a constant depending only on g (see [**Bu2**, §13]). Then using a modified version of an argument of [**B1**] we show that $\lambda_j \cdot (1 + o(\sqrt{\delta})) \geq \pi^{-1} \lambda_j(\Gamma)$, where Γ is the graph of the Y-pieces, and the length of an edge corresponding to a small geodesic is $l(\gamma)$. The proof of this also uses the asymptotic of the first nonzero eigenvalue of $Y_1 \cup Y_2$ for the Neumann problem, where Y_1, Y_2 are Y-pieces, $Y_1 \cap Y_2 = \gamma$ and $l(\gamma)$ is small. This can be deduced from [**C.CdV**], because there is only one small geodesic separating $Y_1 \cup Y_2$. To finish the proof we have then to compare $\lambda_j(\Gamma)$ with $\lambda_j(\Lambda)$. To do this we consider Λ to be the graph of the connected components of Γ after removing the small edges of Γ.

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If f is an eigenfunction corresponding to $\lambda_j(\Gamma)$ we write $f = h + g$, where, on each component, h is constant and g of mean zero: Applying the quadratic form to $h + g$ we estimate the resulting terms.

If one is interested in lower bounds on λ_j in terms of $\lambda_j(\Lambda)$ it is more convenient to relate λ_j to a graph associated to a geodesic triangulation of S as considered in [**Bu1**] and used in [**B2**, §4]. As in the proof of Theorem 1, one relates then the spectrum of this graph to $\lambda_j(\Lambda)$.

THEOREM 2. *Let S be a compact Riemann surface of genus $g \geq 2$ and $l(\gamma_1) \leq \cdots \leq l(\gamma_r) \leq l(\gamma_{r+1}) \cdots$ be the length of the closed geodesics of length smaller than* $2 \ln 2$. *Then there exists a universal constant $c > 0$ s.t. if $l(\gamma_r) \leq l(\gamma_{r+1})/g^2$ then $\lambda_j(S) \geq c\lambda_j(\Lambda)$ for $1 \leq j \leq k$. And Λ is the graph associated to the components of S after removing the geodesics $\gamma_1, \ldots, \gamma_r$.*

REMARKS. (a) It is easy to generalize Theorems 1 and 2 to the case of geometrically finite surfaces.

(b) In [**D.P.R.S.**] the inequality of Theorem 2 is stated with a constant $c(g)$ depending on the genus of S. But for fixed g there are only finitely many graph structures which can occur if one forgets the length of the edges, thus a constant depending on g "destroys" the combinatorial information contained in $\lambda_j(\Lambda)$.

(c) If there are no small geodesics or if the small geodesics don't disconnect the surface then $\lambda_1(S) \geq c/g^2$, $c > 0$ being a universal constant.

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MATHEMATISCHES INSTITUT, RHEINSPRUNG 21, 4051 BASEL, SWITZERLAND