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DAVID A. VOGAN, JR.

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Localization in Noetherian rings, by A. V. Jategaonkar, London Mathematical Society Lecture Note Series, vol. 98, Cambridge University Press, Cambridge, 1986, xii + 324 pp., $34.50. ISBN 0-521-31713-4

The book under review is, to start with, a book about noncommutative Noetherian rings and prime ideals. (We call a ring Noetherian if it satisfies the ascending chain condition on both left and right ideals.) We should first ask why the Noetherian condition and prime ideals are of significance in the noncommutative case. Commutative Noetherian rings arise naturally, for example as the coordinate rings of algebraic varieties, and the prime ideals then correspond to irreducible subvarieties. Noncommutative rings arise most naturally as rings of operators of some sort—some of the first ones studied in this century were rings of differential operators. It is therefore not surprising that the primitive ideals (annihilators of irreducible representations) were studied before prime ideals, and that Artinian rings were studied before Noetherian rings. Jacobson's definitive book of 1956 makes no mention of the Noetherian condition. However, Noetherian rings do arise naturally—certain rings of differential operators are Noetherian, in particular, the enveloping algebras of finite-dimensional Lie algebras. Also, group algebras of polycyclic-by-finite groups are Noetherian. (A group is polycyclic-by-finite if it has a series of normal subgroups such that all of the factors are finite or finitely generated Abelian.) This suggested that the development of a theory making full use of the Noetherian hypothesis should shed particular light on some of the most important examples in representation theory, and recent events have borne this out. The correct definition of prime ideal was pointed out by Krull in 1928 (an ideal P is prime if for any pair of ideals A and B, if $AB \subseteq P$ then $A \subseteq P$ or $B \subseteq P$). (Prime ideals force themselves to one's attention, even if one is only really interested in irreducible representations and primitive ideals.) However, the first important theorem about prime ideals in the Noetherian case was Goldie's theorem, which was not proved until 1958.

What Goldie's theorem provides is the correct analogue in the noncommutative case for the usual field of fractions of a domain in the commutative case. If

(0) is a prime ideal in a Noetherian ring R, we let $\mathscr{C}$ be the set of regular elements of R (i.e., those elements which are neither left nor right zero-divisors). Goldie showed that the set of fractions of the form ac^{-1} ($a \in R$, $c \in \mathscr{C}$) is a ring, which we will write Quo(R), and that this ring of fractions is an $n \times n$ matrix ring over a division ring for some positive integer n. This integer n is the "Goldie rank" of R. The statement that the fractions ac^{-1} form a ring is equivalent (in a Noetherian ring) to the statement that the set $\mathscr{C}$ is a *right Ore set*—that is, that for $a \in R$ and $c \in \mathscr{C}$ there are elements $a' \in R$ and $c' \in \mathscr{C}$ with $ac' = ca'$. (Goldie shows, in fact, that $\mathscr{C}$ is both a right and a left Ore set.) Goldie's work led to the active development of the theory of Noetherian rings. In this review we will concentrate on only one aspect of this development.

Following the example of commutative ring theory, a natural step following Goldie's theorem would be to extend these ideas to localizing at a prime ideal. An initial attack on this problem was made by Goldie in 1967, where for a prime ideal P, he defined $\mathscr{C}(P)$ to be the set of elements in R which are regular modulo P. We say that the prime ideal P is *localizable* if the set $\mathscr{C}(P)$ is a right and left Ore set. Under these circumstances, the corresponding ring of fractions, which we write R_P, will have all the properties one might desire of a localized ring—in particular, PR_P is the only maximal (and only primitive) ideal of R_P, and R_P/PR_P can be identified with Quo(R/P). If prime ideals could be proved to be localizable, one would have a powerful tool in noncommutative algebra. However, it quickly became clear that prime ideals usually are not localizable, and that a localization theory for noncommutative Noetherian rings would either be much more subtle than the commutative theory, or impossible. It now appears that there is indeed a potentially useful localization theory for large classes of Noetherian rings, which is indeed much more subtle than in the commutative case, while at the same time there are many families of Noetherian rings for which no such theory can be expected. This monograph contains a definitive account of the recent work leading to these conclusions, and, in some sense, contains the solution to the localization problem posed by Goldie in 1967.

A key difficulty in localizing at a prime ideal, first pointed out by Jategaonkar in 1973, is that if P and Q are distinct maximal ideals such that R/P and R/Q are Artinian and if $Q \cap P \neq QP$, then P cannot be localizable. In fact, if $\mathscr{C}$ is a right Ore set contained in $\mathscr{C}(P)$, then it is also contained in $\mathscr{C}(Q)$. Jategaonkar suggested that prime ideals should be viewed as tied together by the existence of certain types of ideal factors of R (such as $Q \cap P/QP$ in the above instance), so that the set of prime ideals would break into equivalence classes (hopefully finite). One then might hope to localize primes in batches, as it were. (If $X = \{P_1, \ldots, P_n\}$ is a finite set of incomparable primes, and $S = \bigcap_{1 \leqslant i \leqslant n} P_i$, then the set of elements regular modulo S is just $\bigcap_{1 \leqslant i \leqslant n} \mathscr{C}(P_i)$, and we say that S is a *localizable semiprime ideal* if this set is left and right Ore. This produces a localized ring with many reasonable properties—in particular it has only a finite number of irreducible representations. This, of course, is just the kind of localization you get in the traditional theory of orders when you localize an order over a commutative ring at a prime ideal in the commutative ring.)

In 1976, B. J. Mueller defined a particular kind of tie between two prime ideals—we say there is a *link* from Q to P, written $Q \rightsquigarrow P$, if $Q \cap P/QP$ is large in some suitable sense. For the "fully bounded rings" considered by Mueller, we say $Q \rightsquigarrow P$ if the left and right annihilators of $Q \cap P/QP$ are Q and P respectively. These links make the spectrum of the ring into a graph, and the connected components of this graph are now usually called the *cliques* of prime ideals. Mueller proved that for these rings, if a clique is finite, then the intersection of the primes in the clique is a localizable semiprime ideal, so that there is a reasonable notion of localization for these prime ideals. However, he also showed that the cliques need not be finite. This fact, and the fact that in more general rings it was clear that there were still other obstructions to localizability, seemed to make localization questions of limited interest. Though there were some papers on special cases of localizability, it seemed clear that a general theory with serious applications would not be available. As one expert pointed out, a really good localization theory would provide elegant proofs of many theorems which were already known to be false.

In the meanwhile, starting in the early sixties, other algebraists had tried to develop a much more general notion of localization and rings of quotients, starting from the assumption that fractions were too primitive a concept, and that more general overrings needed to be considered. This line of work had led to the general notion of a "torsion theory" and led to an analysis of overrings (possible localizations) using injective modules. These ideas led to many papers, but to relatively few applications in ring theory. However, the idea that injective modules should hold the key to localization turned out to be important. This idea was taken up by Jategaonkar in the 1970s. In 1982, he introduced a new condition on a Noetherian ring—the *second layer condition*. (A Noetherian ring satisfies the second layer condition if, for every prime ideal P, and every finitely generated submodule M of the injective envelope of R/P whose annihilator is a prime ideal, this prime ideal is actually P.) Though an awkward condition to state, this condition turns out to be a good one in the sense that it is not difficult to check in important cases—enveloping algebras of finite-dimensional solvable Lie algebras and group rings of polycyclic-by-finite groups satisfy the condition; enveloping algebras of semisimple Lie algebras do not. It has already become clear that this condition has important representation-theoretic consequences, and that it will play a role in questions far removed from localization theory.

Using this condition, Jategaonkar was able to find the right general result concerning localization of finite sets of prime ideals. This had remained an interesting question, partly because of potential applications to group rings, and several partial results had been proven. For a Noetherian ring satisfying the second layer condition, a finite set of primes is localizable if and only if it is a union of cliques. (Our definition of link given above is still an adequate one for this class of rings.) While a definitive negative result is not available, it seems implausible that any reasonable localization theory for prime ideals will exist in rings which fail to satisfy the second layer condition. Beginning at about this time, Jategaonkar began work on a monograph on localization theory, in which the above result would presumably have held a central position. However, while the monograph was in preparation, there was a burst of activity which rendered parts of the original monograph obsolete.

The new ingredient which led to this burst of activity in the last few years was the realization that there could indeed be a good theory of localization at infinite sets of primes. If X is a set of incomparable prime ideals, we let $\mathscr{C}(X) = \bigcap_{P \in X} \mathscr{C}(P)$. We say X is *localizable* if (i) $\mathscr{C}(X)$ is a left and right Ore set, and (ii) if R' is the resulting ring of fractions, then the primitive ideals of R' are precisely the ideals PR' for $P \in X$, and for these primes, $R'/PR' \cong \mathrm{Quo}(R/P)$. An initial step was taken by Mueller in 1980, under circumstances which seemed too special to attract general interest. However, two papers of K. A. Brown's published in 1983 and 1984, attracted immediate attention. In these papers, Brown explicitly wrote down all of the cliques for the enveloping algebra of a solvable Lie algebra over the complex numbers, and showed that every clique was localizable in the above sense, thus obtaining a reasonable localization theory for these rings. Progress after this was rapid. Jategaonkar proved a useful abstract localization result for infinite sets of primes, and concrete results for rings satisfying a polynomial identity and for algebras for which all prime factors are domains. J. T. Stafford used his continuity results for functions on $\mathrm{Spec}(R)$ to show that all cliques are either finite or countable. Stafford and the reviewer independently proved that if R is an algebra over an uncountable field and satisfies the second layer condition, and if for every clique X in $\mathrm{Spec}(R)$, there is a bound on the Goldie ranks of the factors R/P for all $P \in X$, then all cliques of R are localizable. K. A. Brown proved that the cliques in the group ring of a polycyclic-by-finite group satisfy this condition on Goldie ranks, thus giving another class of rings for which the localization theory is applicable. All of these results depend on some infinite general position arguments—the significance of Stafford's countability theorem is that having an uncountable field in the center of the ring gives enough room for the general position argument to work. While all of this was taking place, the present monograph was being written, so that anyone trying to go back to the literature to find out just who did what will have difficulties—results in preliminary versions of the monograph are cited in papers, while the new results of those papers are now included in the monograph.

The subject has continued to develop since the monograph reached final form, but the progress has not been so rapid, and the monograph is not only timely, but also is as close to the present state of the subject as is possible in an active area. There have been fewer applications than one might have hoped—the most dramatic being the proof of the long outstanding conjecture that for Noetherian rings satisfying a polynomial identity, the global dimension is bounded below by the Krull dimension (the work of several authors, which appeared in 1984). Since the localization theory concentrates on the two-sided ideal structure of the ring, and so much of what is important in noncommutative rings rests on the one-sided ideal structure, there are inevitable limitations in the applicability of the localization theory. One might therefore expect that the applications would be most striking in situations in which there are many two-sided ideals—e.g., for rings satisfying a polynomial identity (P.I. rings). The progress to date supports this view. A continuing difficulty is that one would like to localize at any clique in a Noetherian P.I. ring, and one does not want to restrict oneself to algebras over uncountable fields. Two recent partial solutions to this problem are Mueller's result that all cliques are localizable in affine Noetherian P.I. algebras, and Stafford's result that if R is a Noetherian P.I. ring and Ω a clique in Spec(R), then the "induced" clique $\Omega[x]$ is always localizable in $R[x]$. It seems likely that we will see more progress along these lines, and more applications as the theory matures.

The present monograph gives a thorough treatment of the localization theory from the author's point of view. It also treats related topics in Noetherian rings, such as Artinian quotient rings, primary decomposition, bimodules over Noetherian rings, and some representation theory of rings satisfying the second layer condition. The material is developed with meticulous care from the very beginning, and in the greatest possible generality. Useful appendices outline the basic theory of four special classes of rings to which the theory applies (hereditary Noetherian prime rings, Noetherian P.I. rings, enveloping algebras of solvable Lie algebras, and group rings of polycyclic-by-finite groups). Its very thoroughness and generality make the monograph not entirely suitable as an introduction, since a novice can easily become lost in the details and the main ideas of the subject are not always clearly emphasized. (It is certainly possible to construct a geodesic from first principles to the main theorems on localization in far fewer pages, largely by omitting side issues. Short introductions to the ideas appear in K. A. Brown's article in the Séminaire d'Algèbre Dubreil-Malliavin (Springer-Verlag, Lecture Notes in Math., vol. 1146, 1986, pp. 355–366), in A. Bell's unpublished notes on localization theory, and in a chapter in the forthcoming book on Noetherian rings by McConnell and Robson.) On the other hand, for the professional the work is likely to be indispensable. In addition to the usual feature of collecting material otherwise scattered through the literature, the monograph contains a great deal of unpublished material, much of it due to the author. A copy should certainly be on the shelf of anyone seriously working on Noetherian rings.

ROBERT B. WARFIELD