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A. E. HURD

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A local spectral theorem for closed operators, by Ivan Erdelyi and Wang Shengwang. London Mathematical Society Lecture Notes Series, vol. 105, Cambridge University Press, Cambridge, London, New York, New Rochelle, Melbourne, Sydney, 1985, \$22.95. ISBN 0-521-31314-7

Spectral theory has its origin in the theories of matrices and integral equations, and in the case of unbounded operators, in the theory of differential equations. David Hilbert was the first mathematician to use the word "spectrum" in its present meaning: if T is a bounded operator acting in a (complex) Banach space X , its spectrum $\sigma(T)$ is the set of all complex numbers λ , for which $\lambda I - T$ is singular (that is, not invertible in the Banach algebra $B(X)$ of all bounded linear operators on X). A central role in the study of the spectrum is played by the resolvent operator $R(\lambda; T) = (\lambda I - T)^{-1}$, defined and analytic in the resolvent set $\rho(T) = \mathbb{C} \setminus \sigma(T)$, and by the analytic operational calculus $\tau: f \rightarrow f(T)$, which is a continuous homomorphism of the topological algebra $H(\sigma(T))$ of all functions analytic in a neighborhood of $\sigma(T)$ into $B(X)$, such that $\tau(f_0) = T$ for $f_0(\lambda) \equiv \lambda$. The theory of analytic functions became a powerful tool in spectral theory in the work of Laguerre (1867), Frobenius (1896), and Poincaré (1899), and later in the now classical work of F. Riesz, Hilbert, Wiener, Stone, Beurling, Gelfand, Dunford, and many others. The related spectral theory of Banach algebras, pioneered by Gelfand and Shilov, follows a similar path. For unbounded closed operators, Taylor extended the Riesz-Dunford integral formula for $\tau(f)$ in the form

$$\tau(f) = f(\infty) + \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R(\lambda; T) d\lambda$$

for f analytic in a neighborhood of $\sigma(T) \cup \{\infty\}$ (and Γ a suitable contour).

The analytic operational calculus leads to the construction of projections $E(\sigma)$ associated with the spectral sets σ of T (i.e., the open-closed subsets of $\sigma(T)$ in the relative topology), by means of the formula

$$E(\sigma) = \frac{1}{2\pi i} \int_{\Gamma(\sigma)} R(\lambda; T) d\lambda$$

with a suitable contour $\Gamma(\sigma)$. This can be used to obtain the classical Jordan reduction theory for operators in \mathbb{C}^n . Later, the method was refined by Dunford to obtain the Riesz spectral theory of compact operators (which generalizes Fredholm's theory of integral operators).

The elementary observation that the map $\sigma \rightarrow E(\sigma)$ is a homomorphism of the Boolean algebra of spectral sets onto a Boolean algebra of projections in X (which takes the "unit" $\sigma(T)$ onto I), leads to another central concept of spectral theory, namely that of a *spectral measure*. If \mathcal{B} denotes the Boolean algebra of all Borel sets in \mathbb{C} , a spectral measure on \mathcal{B} is an algebra homomorphism $E(\cdot): \mathcal{B} \rightarrow B(X)$ such that $E(\sigma(T)) = I$ and $E(\cdot)x$ is countably additive for each $x \in X$. If, in addition, $E(\delta)$ commutes with T for each $\delta \in \mathcal{B}$, and the restrictions $T|E(\delta)X$ have their spectra in the closure $\bar{\delta}$ of δ (for each $\delta \in \mathcal{B}$), the study of T may be reduced to an analysis of operators with smaller spectrum. This important idea is due to Dunford (1951–1954), who called operators T which possess such a spectral measure "*spectral operators*". The corresponding spectral measure is unique, and is called *the resolution of the identity* for T . A spectral operator T relates to its resolution of the identity $E(\cdot)$ by means of a Jordan-type decomposition

$$T = \int_{\sigma(T)} \lambda dE + N,$$

where N is a generalized nilpotent operator ($\|N^k\|^{1/k} \rightarrow 0$) commuting with E . T is said to be of scalar type if $N = 0$. The content of the classical spectral theorem for normal operators in Hilbert space is that every T commuting with its adjoint T^* is spectral of scalar type. The original proof of the spectral theorem for bounded self-adjoint operators ($T = T^*$) goes back to Hilbert (1906), and was later modernized by F. Riesz (1913), and extended to unbounded operators by von Neumann (1929). Since then, many proofs have been found, with methods ranging from analytic function theory to the moments problem to the Gelfand-Naimark theory of commutative B^* -algebras, etc.

The central problem of the theory of spectral operators is that of finding easily applicable characterizations. Dunford's original approach to the characterization problem can be seen as the point of departure of what the present monograph calls the "local spectral theory" of operators. A necessary property of spectral operators is the so-called "single-valued extensions property" (SVEP). Stated for an unbounded operator T with domain $D(T)$, the SVEP means that whenever f is a $D(T)$ -valued function analytic in some open set, such that $(\lambda I - T)f(\lambda) \equiv 0$, then $f(\lambda) \equiv 0$. This implies that there exists a $D(T)$ -valued analytic function $x(\cdot)$ with maximal open domain $\rho(x)$ such that $(\lambda I - T)x(\lambda) \equiv x$. The function $x(\cdot)$ is called the "local resolvent" of T at x ; $\rho(x)$ is the "local resolvent set," and its complement $\sigma(x)$ is the "local spectrum." For each $F \subset \mathbb{C}$, the set

$$X(F) = \{x \in X \mid \sigma(x) \subset F\}$$

is a linear manifold hyperinvariant for T , called the "spectral manifold" for T . A second necessary property of a spectral operator T is that the spectral

manifolds $X(F)$ are closed for closed F (Dunford's Condition (C)). Consequently, for closed F , $X(F)$ is a reducing subspace for T such that

$$\sigma(T|X(F)) \subset F \cap \sigma(T).$$

These "local" concepts are analyzed and refined in Chapter 1 of the present monograph. Chapter 2 deals with the *decomposability* of operators. When Dunford's Condition (C) holds, $X(F)$ is a typical "spectral maximal space" for T (for F closed), that is, an invariant subspace Y with the "maximality" property that whenever Z is an invariant subspace with $\sigma(T|Z) \subset \sigma(T|Y)$, then $Z \subset Y$. Actually, each spectral maximal space Y has the form $X(\sigma(T|Y))$. The operator T is said to have the spectral decomposition property (SDP) if for any finite open cover $\{G_i\}_{i=0}^n$ of $\sigma(T)$ with $\infty \in G_0$ and \bar{G}_i compact for $i \geq 1$, there exist invariant subspaces X_i ($i = 0, \dots, n$) such that $X_i \subset D(T)$ ($i = 1, \dots, n$), $\sigma(T|X_i) \subset G_i$ ($i = 0, \dots, n$), and $X = \sum_{i=0}^n X_i$. In case the X_i can be chosen as spectral maximal spaces for T , then T is said to be *decomposable*. T is decomposable iff it has the SDP and $D(T) \supset X(F)$ for some compact F (in particular, for T bounded, T is decomposable iff it has the SDP). The theory of *bounded* decomposable operators has been extensively studied by Colojoară and Foiaş (1968) and their followers, mostly of the Romanian school of operator theory. The theory has a smooth generalization to unbounded closed operators, thanks to the fact that T has the SDP iff $R(\lambda; T)$ has the SDP (i.e., is decomposable), for $\lambda \in \rho(T)$ fixed.

For a densely defined closed operator T , it is shown (in Chapter 3) that T has the SDP iff T^* has the SDP. More detailed results along these lines are found in Chapters 4 and 5.

The monograph is essentially self-contained, and is a useful and well-written exposition of some of the most recent aspects of spectral theory, to which the authors have significantly contributed since 1975. The bibliography on the specific topics of the book is probably complete up to 1984. In addition to the index, a comprehensive glossary of notations is provided.

Together with the books by Dunford and Schwartz, Dowson, and Colojoară and Foiaş, the monograph under review brings today's researcher in the field to some of its interesting frontiers.

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SHMUEL KANTOROVITZ