The book contains much more than the mere proof of the two principal results. In the first two chapters Gromov explains basic facts about manifolds of nonpositive curvature and of course does this in his very own original style. Therefore this introduction contains exciting new ideas even for experts. Certainly the most important one is the definition of the Tits metric for points at infinity of a complete simply connected manifold of nonpositive curvature. If x and y are such points then $\sphericalangle(x, y)$ is defined as the supremum of all angles under which x and y can be seen from a finite point. The Tits distance Td is then defined as the corresponding inner metric, i.e. as the infimum of the length (w.r.t. $\sphericalangle$) of all curves in the boundary connecting the points. One of its main features is that flat totally geodesic subspaces are reflected by this metric. They give rise to isometrically embedded round spheres in the boundary. In particular, the boundary of $\mathbf{R}^n$ with the Tits metric is the standard sphere S^{n-1}. For a symmetric space, Tits metric and Tits building contain the same information and determine each other. This explains the name. There is no doubt that the Tits metric will be an important tool for further investigations.

The book by M. Gromov, V. Schroeder, and W. Ballmann is an extraordinary one which contains a wealth of new ideas as well as plenty of inspiration for further work. (For example the study of the relation between Tits metric and fundamental groups of compact quotients seems to be very promising.) It is of course not a textbook in the usual sense. Although it starts quite easy it ends more or less like a research paper. But due to the excellent work of V. Schroeder the presentation is always clear. I think the book will have a strong influence on the further development of the theory of manifolds of nonpositive curvature.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 17, Number 2, October 1987

0273-0979/87 $1.00 + $.25 per page

Diophantine inequalities, by R. C. Baker, London Mathematical Society Monographs, New Series, vol. 1, Clarendon Press, Oxford, 1986, xii + 275 pp., $65.00. ISBN 0-19-853545-7

The volume under review deals with analytic methods for diophantine inequalities. For diophantine inequalities, and in fact for many diophantine problems in general, the appropriate tool is Fourier analysis. Among the most important situations are (i) when the underlying group $G = \mathbf{Z}$, with the dual group $\hat{G}$ the torus $T = \mathbf{R}/\mathbf{Z}$, (ii) when $G = T$, $\hat{G} = \mathbf{Z}$, (iii) when $G = \hat{G} = \mathbf{R}$, and (iv) when $G = \hat{G} = \mathbf{Z}/m\mathbf{Z}$.

The Hardy-Littlewood circle method rests on Fourier analysis on $\mathbf{Z}$, so that we are in case (i). Writing $e(z) = e^{2\pi i z}$, we have $\int_T e(n\alpha)\,d\alpha = 1$ or 0 when $n = 0$ or $n \in \mathbf{Z} \setminus 0$, respectively. Thus given a polynomial $P(\mathbf{x}) = P(x_1, \ldots, x_s)$ with integer coefficients, and given a bounded domain $\mathscr{D} \subset \mathbf{R}^s$, the number $Z = Z(P, \mathscr{D})$ of integer points $\mathbf{x} \in \mathscr{D}$ with $P(\mathbf{x}) = 0$ is given by

$$Z = \int_T f(\alpha)\,d\alpha \tag{1}$$

with

$$f(\alpha) = \sum_{\mathbf{x} \in \mathscr{D}} e(\alpha P(\mathbf{x})). \tag{2}$$

In practice, the method depends on a judicious partition of T into intervals, into the so-called major and minor arcs, in order to estimate the integral in (1). An excellent account of the method may be found in a recent book of Vaughan [**8**]. The best known application of the circle method is on Waring's Problem, i.e. the problem of representing integers n as sums of nonnegative kth powers:

$$n = x_1^k + \cdots + x_s^k. \tag{3}$$

Here (1), (2) may be applied with $P = x_1^k + \cdots + x_s^k - n$, and $\mathscr{D}$ the domain $0 \leqslant x_i \leqslant n^{1/k}$ $(i = 1, \ldots, s)$. Weyl's estimate for exponential sums leads relatively quickly to $G(k) \leqslant 2^k + 1$, where $G(k)$ is the least value of s such that (3) may be solved for sufficiently large n, say for $n \geqslant n_1(k)$. The more sophisticated Vinogradov estimate leads to better results when k is large, in particular to $G(k) < c_1 k \log k$, with an absolute constant c_1. The method is much more difficult to apply to problems which, in contrast to Waring's problem, are not of additive type. In this direction, the most outstanding achievements were obtained by Davenport [**3**] and by Heath-Brown [**5**] on cubic equations.

In the present book, whose content is entirely different from that of Vaughan, diophantine problems on all the four groups mentioned above are studied, with emphasis on (ii), (iii), (iv). These problems, which are in general less well known than the diophantine equations treated by the Hardy-Littlewood method, have a character and charm of their own. The methods and results are sometimes parallel to the ones in the case (i), but very often extra difficulties arise. We are still dealing with problems where the variables are integers or integer s-tuples, but instead of polynomials P with integer coefficients, we have polynomials with coefficients (and therefore values) in a group G.

Let us begin with the case (ii), so that $G = T$. Given $\xi \in T$, say ξ the coset of an element $\alpha \in \mathbf{R}$ in $\mathbf{R}/\mathbf{Z} = T$, let $\|\xi\|$ denote the distance from α to the nearest integer. Now if $P(\mathbf{x})$ is a polynomial with coefficients in T, and if $\mathscr{D} \subset \mathbf{R}^s$, we are interested in the number $R = R(P, \mathscr{D}, \delta)$ of integer points

$\mathbf{x} \in \mathscr{D}$ which are solutions of the diophantine inequality

$$\|P(\mathbf{x})\| < \delta.$$

The simplest idea would be to work with the function $\phi: T \to \mathbf{R}$ which is 1 when $\|\xi\| < \delta$, and 0 otherwise. Since ϕ has no nice Fourier expansion (even if we redefine $\phi(\xi) = \frac{1}{2}$ for ξ with $\|\xi\| = \delta$), it is better to work with a smooth function ϕ^* which is close to ϕ and whose Fourier expansion converges rapidly. Suitable functions ϕ^* were introduced by Vinogradov, and by Beurling and Selberg. If, say, $\phi^* \leqslant \phi$, then

$$R \geqslant \sum_{\mathbf{x} \in \mathscr{D}} \phi^*(P(\mathbf{x})).$$

Given the Fourier expansion $\phi^*(\xi) = \sum_{\nu \in \mathbf{Z}} a_\nu e(\nu\xi)$, we obtain

$$R \geqslant \sum_{\nu \in \mathbf{Z}} a_\nu f(\nu) \tag{4}$$

with

$$f(\nu) = \sum_{\mathbf{x} \in \mathscr{D}} (\nu P(\mathbf{x})), \tag{5}$$

in analogy with (1), (2).

There is usually no need to apply Fourier analysis to linear problems. Dirichlet's Theorem says that given $\alpha \in \mathbf{R}$ and $N > 1$, there are integers x, y with $1 \leqslant x \leqslant N$ and $|\alpha - (y/x)| < 1/(Nx)$, i.e. with $|\alpha x - y| < N^{-1}$. Thus given $\alpha \in T$ and $N > 1$, there is a natural $x \leqslant N$ with $\|\alpha x\| < N^{-1}$. Considerable difficulties arise if we try to approximate $\alpha \in \mathbf{R}$ by fractions of the type y/x^2, i.e. if we try to make $\|\alpha x^2\|$ small. Heilbronn **[6]**, improving on earlier work of Vinogradov, used the method outlined above to prove the existence of an x in $1 \leqslant x \leqslant N$ with $\|\alpha x^2\| < c_2(\varepsilon)N^{-(1/2)+\varepsilon}$. More generally, given natural k, there is an x in $1 \leqslant x \leqslant N$ with $\|\alpha x^k\| < c_2(k, \varepsilon)N^{-(1/K)+\varepsilon}$, where $K = 2^{k-1}$. The exponent here is almost certainly not best possible for $k > 1$, and it may be conjectured that the correct exponent is $-1 + \varepsilon$. For small values of $k > 1$ the exponent $-(1/K) + \varepsilon$ appears to be extremely difficult to improve, and few mathematicians expect a big improvement soon. On the other hand, it had been annoying for some time that for polynomials $P(x) = \alpha_k x^k + \cdots + \alpha_1 x$ in place of αx^k, only an exponent even worse than $-(1/K) + \varepsilon$ could be established. R. C. Baker, in technically brilliant work **[1]**, ("solution of Davenport's problem") did in fact obtain the exponent $(-1/K) + \varepsilon$ for general P with constant term zero. All this is achieved using Weyl type estimates, but for large values of k Vinogradov's estimate yields the exponent $-1/K'$ with $K' = c_3 k^2 \log k$. This is in analogy with the two types of estimates for $G(k)$ in Waring's Problem.

The results mentioned so far are contained in Chapters 2–6 of the work under review. Simultaneous approximations are treated in Chapters 7 and 8. Dirichlet's Theorem say that for $\alpha_1, \ldots, \alpha_h$ in T and for $N > 1$, there is an x in $1 \leqslant x \leqslant N$ with $\|\alpha_i x\| < N^{-1/h}$ $(i = 1, \ldots, h)$; this is best possible. As a simultaneous approximation version of Heilbronn's Theorem, it may be shown that there is a natural $x \leqslant N$ with

$$\|\alpha_i x^2\| < c_4(h, \varepsilon)N^{-1/(h^2+h)+\varepsilon} \qquad (i = 1, \ldots, h).$$

Similar, but weaker, results may be established with $\alpha_i x^2$ replaced by polynomials $P_i(x)$ of degree k and with zero constant term. The results may be interpreted geometrically, and the proofs depend on a "lattice method" from the geometry of numbers in conjunction with the usual Fourier transform.

In Chapters 9 and 10, quadratic forms in several variables, as well as additive forms, are considered. For example, given real quadratic forms $Q_1, \dots, Q_h$ in s variables, and given N, there exist integer points $\mathbf{x}$ with maximum norm $|\mathbf{x}| \leqslant N$ and

$$(6) \qquad \|Q_i(\mathbf{x})\| < N^{-(2/h)+\varepsilon} \qquad (i = 1, \dots, h),$$

provided that $s > s_1(h, \varepsilon)$. The exponent is essentially best possible. This is a typical "large number of variables" result, for the numbers $s_1 = s_1(h, \varepsilon)$ obtainable (but no person in his/her right mind would actually try to compute it) by present methods is huge. No precise analogue of (6) is known for forms of even degree $k > 2$. For forms of odd degree d, there is a stronger result with arbitrary exponent $-E$, derived in Chapter 14.

Now let us turn to the case (iii) with $G = \mathbf{R}$. Given a polynomial $P(\mathbf{x})$ with real coefficients and a domain $\mathscr{D}$, we write $S = S(P, \mathscr{D}, \delta)$ for the number of integers points $\mathbf{x} \in \mathscr{D}$ with

$$|P(\mathbf{x})| < \delta.$$

With

$$\psi^*(\xi) = \max\left(0, 1 - |\xi/\delta|\right) = \delta^{-1} \int_{\mathbf{R}} e(\alpha\xi)\left(\frac{\sin \pi\alpha\delta}{\pi\alpha}\right)^2 d\alpha,$$

we have

$$S \geqslant \sum_{\mathbf{x} \in \mathscr{D}} \psi^*(P(\mathbf{x})),$$

and therefore

$$(7) \qquad S \geqslant \delta^{-1} \int_{\mathbf{R}} \left(\frac{\sin \pi\alpha\delta}{\pi\alpha}\right)^2 f(\alpha)\, d\alpha,$$

with f given by (2). Davenport and Heilbronn [4] used this approach to show that an inequality

$$\left|\alpha_1 x_1^2 + \cdots + \alpha_5 x_5^2\right| < \delta$$

with real $\alpha_1, \dots, \alpha_5$, not all of the same sign, has a nontrivial solution. The same conclusion was later shown to be true for a general indefinite quadratic form in at least 21 variables. One of the outstanding open problems is to show that 5 variables suffice, and perhaps 3 variables suffice if the coefficients are not all in rational ratios. (*Note added in proof*. G. A. Margulis dealt with this in a recent manuscript, "Indefinite quadratic forms and unipotent flows on homogeneous spaces".) This topic is not explored in the present treatise, but in Chapter 14 one finds the following "many-variable" result: given forms $P_1, \dots, P_h$ of odd degree k with real coefficients, the simultaneous inequalities

$$(8) \qquad |P_i(\mathbf{x})| < \delta \qquad (i = 1, \dots, h)$$

have a nontrivial integer solution, provided the number of variables exceeds $s_2 = s_2(h, k)$. This contains a well-known theorem of Birch [**2**] on systems of diophantine equations, and in fact the proof depends on a quantitative version of Birch's theorem, which is derived in Chapters 11 and 13 by an adaptation of the circle method.

The situation (iv) with $G = \hat{G} = \mathbf{Z}/m\mathbf{Z}$ is in principle the simplest one. When $\mathscr{D} \subset (\mathbf{Z}/m\mathbf{Z})^s$, the number $T = T(P, \mathscr{D})$ of $\mathbf{x} \in \mathscr{D}$ with $P(\mathbf{x}) \equiv 0 \pmod m$ is given by

$$T = m^{-1} \sum_{a \in \mathbf{Z}/m\mathbf{Z}} f(a) \tag{9}$$

where

$$f(a) = \sum_{\mathbf{x} \in \mathscr{D}} e\left(\frac{a}{m} P(\mathbf{x})\right). \tag{10}$$

In practice, this method depends on estimates for exponential sums in finite fields, such as the ones derived by Weil and by Deligne, as well as a recent one by this reviewer presented in Chapters 15, 16 and 17.

Much work has recently been done on "small" solutions of congruences. In Chapter 9 one finds Heath-Brown's result that a quadratic congruence $Q(x_1, x_2, x_3, x_4) \equiv 0 \pmod p$ to a prime modulus has a solution $\mathbf{x} \neq \mathbf{0}$ with $|\mathbf{x}| \leqslant c_5 p^{1/2} \log p$, but derived with Fourier analysis on $\mathbf{Z}$ rather than $\mathbf{Z}/p\mathbf{Z}$. "Large number of variables" theorems are derived in Chapters 12 and 18. For example, if $P_i(\mathbf{x}) = a_{i1}x_1^k + \cdots + a_{is}x_s^k$ $(i = 1, \ldots, h)$ are additive forms, then the congruences $P_1(\mathbf{x}) \equiv \cdots \equiv P_h(\mathbf{x}) \equiv 0 \pmod m$ have a solution $\mathbf{x} \neq \mathbf{0}$ with $|\mathbf{x}| \leqslant m^{(1/k)+\varepsilon}$, provided that $s \geqslant s_4(h, k, \varepsilon)$. But when $P_1, \ldots, P_h$ are arbitrary forms of degree k, then the system of congruences has a solution $\mathbf{x} \neq \mathbf{0}$ with $|\mathbf{x}| \leqslant m^{(1/2)+\varepsilon}$ when $s \geqslant s_5(h, k, \varepsilon)$. It would be of great interest to have reasonable bounds on the required number of variables; some progress on this has been made in recent work [**7**].

The technical difficulties in this area are rather formidable. The author presents a wealth of material, and he carries out the proofs with the necessary details. He gives an exposition of the anything but simple Vinogradov method, he presents his own highly technical "solution to Davenport's problem," and he is brave enough to devote three chapters toward the end to exponential sum estimates. There is a danger that readers may be turned off by highly complex proofs which sometimes run through several chapters, and thus the exposition may not gain as many friends for the subject as it deserves. On the other hand, the author has done a magnificent job presenting so much material so thoroughly.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 17, Number 2, October 1987

0273-0979/87 $1.00 + $.25 per page

Nonstandard methods in stochastic analysis and mathematical physics, by Sergio Albeverio, Raphael Høegh-Krohn, Jens Erik Fenstad, and Tom Lindstrøm, Academic Press, Orlando, xi + 514 pp., $59.50 cloth, $39.50 paperback. ISBN 0-12-048860-4

Nonstandard or infinitesimal analysis was invented by the late Abraham Robinson in 1960. Since that time there has been continued interest in the subject and a number of impressive results have been established using nonstandard methods. These results testify to the vision of the man of whom Gödel wrote, "(He was) the one mathematical logician who accomplished incomparably more than anybody else in making this science fruitful for mathematics. I am sure his name will be remembered by mathematicians for centuries." The book under review is a welcome addition to a growing list devoted to the subject.

Nonstandard analysis has had a controversial history. It had its roots in the use of infinitesimals by Leibniz and Newton in the development of calculus. Infinitesimals are "numbers" which are smaller in absolute value than any real number. Leibniz regarded them as entities in some "ideal" structure which also contained the infinitely large numbers and the reals. He also implicitly made the important but somewhat vague hypothesis that this structure satisfied the same rules as the ordinary real number system. The challenge facing Robinson was thus to

(a) demonstrate the existence of a set $^*R$, now called the hyperreal numbers, which carried analogues of all the structures on the reals R (for example, the ring and the set theoretic structures);

(b) ensure that statements true in the real number system are mirrored in a natural way by statements true in the structures on $^*R$.