BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 17, Number 2, October 1987

0273-0979/87 $1.00 + $.25 per page

Asymptotics of high order differential equations, by R. B. Paris and A. D. Wood. Pitman Research Notes in Mathematics, vol. 124, Longman Scientific and Technical, Essex, and John Wiley, New York, 344 pp., $49.95. ISBN 0-470-20375-7

The theory of differential equations is an eminently applicable branch of mathematics. The differential equations that govern physical phenomena are mostly too complicated to allow a complete mathematical analysis. Usually, they involve partial derivatives of order higher than two and are nonlinear. The first part of the efforts at solving them then consists in drastic simplifications by assumptions that narrow down severely the scope of the results. Naturally, the simpler the new differential equations the more one can say about them. It is true that in the last few decades the development of fast computing machines has widened the range of problems that can be approached by numerical analysis, and, indirectly, the many unexpected phenomena discovered by numerical experiments have stimulated the interest in nonlinear equations.

There is, however, still much to be done in the long-studied theories of quite simple differential equations. Even for ordinary linear differential equations with analytic coefficients there exist more unanswered interesting questions—both from the theoretical point of view as well as for the applications—than most outsiders to the field realize. The simplest nontrivial such differential equations, the ones of order two, have been thoroughly explored, particularly in the nineteenth century. This literature fills many shelves, and even dry summaries of results with a few numerical tables added require several volumes.

The book under discussion goes beyond this classical material in that it concentrates on differential equations of order greater than two. It is not especially concerned with equations whose order is a *large* integer, although its title might suggest such a misunderstanding. In fact, the illustrative examples included—and I am glad that there are so many—are mostly of order not exceeding six. The differential equations are assumed to be ordinary, linear and homogeneous, and their coefficients are very special, simple analytic functions.

If these equations are written in the form

$$u^{(n)} - \sum_{r=0}^{p} c_r(z) u^{(r)} = 0, \tag{1}$$

where $0 \leqslant p \leqslant n$, the c_r are analytic functions of the complex variable z, and $u^{(r)} = d^r u/dz^r$, then a basic, very reassuring theorem guarantees that in a neighborhood of any point z_0 at which all c_r are holomorphic (i.e., regular analytic) all solutions u are also holomorphic. Therefore, they can be represented there by convergent series in nonnegative powers of $z - z_0$, and the coefficients of these series can be successively calculated. If a *global* understanding of the solutions is desired, their study near the singularities of the functions c_r in (1) is essential. This study is what the title of the book refers to by the word "asymptotics".

Now, the methods used in the asymptotic theory of linear ordinary analytic differential equations can be divided into two parts: One is based on expansions into power series—usually divergent ones—the other makes use of integral transformations and represents the solutions as definite integrals with respect to a parameter. For a global theory, a two-pronged attack involving both approaches is necessary, except in the simplest cases.

The power series method has the advantage that it leads to an algorithm by means of which any differential equation of the form (1) can be solved near poles of the coefficients c_r. This extends even to systems of differential equations. It has, however, two great shortcomings: One is that the power series appearing in the solutions are divergent, except for a special class of singularities quaintly called "regular singularities". In the happy early days of mathematical analysis, when it was taken for granted that any plausible formalism for the solution of a problem was logically valid, this aspect was not given much thought; but when the concern with rigorous reasoning began to penetrate calculus, divergent power series were for a while regarded as meaningless blind alleys. Only after the middle of the 19th century was it recognized that the partial sums of these series are approximations to solutions in the sense that their kth partial sums differ from a solution by an error of magnitude $o[(z - z_0)^k]$, if z_0 is the singularity of the differential equation under consideration. This is a weaker assertion than to say that the series converges, and this weakening is connected with the second shortcoming of the representation by divergent power series: The solutions represented in this manner, as z approaches z_0, are not the same for approach from all directions. In different sectors the same formal series may represent different solutions. This seemingly paradoxical phenomenon caused considerable surprise and confusion when it was discovered. The difficult task of finding the relations between these solutions has been called the "lateral connection problem". Considerably more difficult still is the "central connection problem", which is to find the convergent power series expansions for the solutions near an ordinary, not singular, point from their asymptotic, divergent representations near a singular point z_0.

The book being reviewed strongly favors the other approach, the one based on integral transformations and the resulting representation of solutions by definite integrals. The great advantage of this method is that such integrals represent the solution wherever they converge, which, in general, is a large region of the complex plane. As there is no free lunch, the applicability of these methods is severely restricted by the sad fact that the transformed differential equation is usually much more complicated than the given one. For this reason the coefficients c_r of the differential equation dealt with in the book are rather special, very simple functions. Almost the whole treatise deals with equations of the form

$$(2) \qquad u^{(n)} = z^{\beta} \sum_{r=0}^{p} a_r z^r u^{(r)}, \qquad n \geqslant p \geqslant 0,$$

with real β and with complex constants a_r. In addition, one chapter is devoted to the more general differential equations

$$(3) \qquad u^{(n)} = \sum_{r=0}^{p} a_r z^{b_r} u^{(r)} \quad \text{with real } b_r,$$

and

$$(4) \qquad u^{(n)} = z^{\beta} \sum_{r=0}^{p} p_r(z) u^{(r)}, \quad (p_r \text{ a polynomial of degree } r).$$

Here the two ways of attacking the problem, as explained above, must be combined, but much of the work is formal, without complete proofs.

The simplest and oldest integral transformations are those named after Fourier and after Laplace. For analytic functions they are best considered as special cases of transformations with the kernel $\exp(sz)$ and with varying paths of integration in the complex plane. Many simple linear differential equations, particularly those with linear coefficients, can conveniently be solved in this manner. The bulk of the book under review is, however, based on the Mellin-Barnes transformations, which are defined by the integral

$$(5) \qquad u(z) = \frac{1}{2\pi i} \int_C g(s) z^s \, ds$$

with a suitable contour of integration C. Although it looks like a very close relative of the complex Laplace transformation, it turns out to be a remarkably flexible tool for the solution of differential equations of the type considered. Indeed, almost all differential equations of the type in formula (2) possess fundamental systems of solutions of the form (5) with explicitly known functions g.

While this method is powerful and fairly general, the formulas that describe it are so involved and long that a brief review cannot enter into any details. It must suffice here to state that for the class of functions represented by Mellin-Barnes integrals in this book g is of the form

$$(6) \qquad g(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1 + s - a_j)}{\prod_{j=m+1}^{q} \Gamma(1 + s - b_j) \prod_{j=n+1}^{p} \Gamma(a_j - s)}$$

(the constants in this formula are not the constants with the same name in formulas (2) and (3)). The functions u defined by (5) when g is in the class (6) are called Meijer G-functions. In an introductory chapter the authors supply a self-contained exposition of the known properties of these G-functions. After that the book gives a detailed account of the use of the Meijer G-functions for the solution of the differential equations. Most of that work is due to the authors. It involves very lengthy and involved manipulations, but the explanations are explicit enough to make verification by the reader possible.

Other representations of solutions are mentioned, for instance those in terms of generalized hypergeometric functions. The G-functions have one decisive advantage over these alternatives: The differential equations have irregular singularities at infinity. There, different solutions may have different orders of magnitude. In linear combinations of these solutions the fastest-growing component present in them determines their order of magnitude. In the solutions by G-functions the several orders of magnitude appear separately, while in the solutions by generalized hypergeometric functions these distinctions are obliterated by the presence of a contribution from the most dominant solution.

The last third of the book deals with applications. Some of them originate in physics, such as boundary layers in magnetohydrodynamics, plasmas, stellar winds and viscous flows. Other applications are purely mathematical. They concern the spectral theory of differential operators in Hilbert space, in particular the extension to higher order of differentiation of the Titchmarsh-Weyl theories on the existence and number of L^2-eigenfunctions.

I believe that this book will be often useful to readers who are looking for ways to deal with some particular differential equation of order higher than two. Rarely will it be studied from beginning to end. The task of following a thread through the book to the formulas and techniques needed in the study of some specific equation would have been made easier if the displayed formulas had been printed in a more easily readable type.

It is often possible to construct transformations that reduce differential equations of a general class into equations of the special forms analyzed in this book. This "comparison" technique is well developed for second-order equations. I would be pleased if this book stimulated the search for more general transformations of this kind.

WOLFGANG WASOW

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Manifolds of nonpositive curvature, by Werner Ballmann, Mikhael Gromov and Viktor Schroeder. Progress in Mathematics, vol. 61, Birkhäuser, Boston, Basel, Stuttgart, 1985, iv + 263 pp., $37.00. ISBN 0-8176-3181-X

This book grew out of four lectures delivered by Mikhael Gromov in 1981 at the College de France in Paris. Its purpose is twofold, namely to give an introduction to manifolds of nonpositive curvature and to give the proof of two outstanding results: the rigidity of locally symmetric spaces in the class of all manifolds of nonpositive curvature (in generalization of Mostow's rigidity theorem), as well as an estimate for the topology of nonpositively curved *analytic* manifolds of finite volume (for more precise statements see below).