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<div align="right">GEORGE E. ANDREWS</div>

One-parameter semigroups of positive operators, by W. Arendt, A. Grabosch, G. Greiner, U. Groh, H. P. Lotz, U. Moustakas, R. Nagel, F. Neubrander, and U. Schlotterbeck. Edited by R. Nagel. Lecture Notes in Mathematics, vol. 1184, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1986, x + 460 pp., $36.30. ISBN 3-540-16454-5

A *strongly continuous semigroup* (or (C_0) *semigroup*) is a family $T = \{T(t) : t \geqslant 0\}$ of bounded linear operators on a Banach space X satisfying, for all s, $t \in [0, \infty)$ and all $f \in X$,

$$T(s + t)f = T(t)T(s)f, \qquad T(0)f = f,$$

$$T(\cdot)f \colon [0, \infty) \to X \text{ is continuous.}$$

The *(infinitesimal) generator* A of T is the strong derivative of T at the origin. More precisely, $f \in \mathrm{Dom}(A)$ and $Af = g$ means $\lim_{t \to 0} t^{-1}(T(t)f - f)$ exists and equals g. Informally one thinks of $T(t)$ as $\exp(tA)$, but care must be exercised in the interpretation of the exponential because in all the interesting cases the generator A is an unbounded operator.

Associated with the generator A (or more generally with a given linear operator A) is the initial value problem

$$(1) \qquad du(t)/dt = Au(t), \qquad u(0) = f \in \mathrm{Dom}(A).$$

The obvious candidate for the solution is $u(t) = T(t)f$ (with the semigroup property $T(t + s)f = T(t)T(s)t$ following formally from the existence and uniqueness for (1)). Of concern is when (1) is a *well-posed problem*. This means that a solution of (1) exists, is unique, and depends continuously on the ingredients of the problem (namely f and A) in a suitable sense.

Two principal results of semigroup theory go back to E. Hille, K. Yosida, and R. Phillips, and can be stated as follows. (I) The initial value problem (1) is well posed iff A is the generator of a (C_0) semigroup T, in which case $u(\cdot) = T(\cdot)f$ is the unique solution of (1). (II) The operator A is the generator of a semigroup iff for λ real and large, $(\lambda - A)^{-1}$ exists in $\mathscr{L}(X)$ (i.e., as a bounded linear operator on X) and certain norm estimates hold.

The resolvent operator is obtained formally from the semigroup via

$$(2) \qquad\qquad (\lambda - A)^{-1}f = \int_0^\infty e^{-\lambda t}T(t)f\,dt,$$

so finding T from $\{(\lambda - A)^{-1}\}$ reduces to inverting a Laplace transform. To recover T from $\{(\lambda - A)^{-1}\}$, the exponential formula

$$(3) \qquad T(t)f = \lim_{n\to\infty}\left(I - \frac{t}{n}A\right)^{-n}f = \lim_{n\to\infty}\left[\frac{n}{t}\left(\frac{n}{t} - A\right)^{-1}\right]^n f$$

is always valid for (C_0) semigroups.

Linear initial value problems for partial differential equations can often be put into the form (1). This involves choosing a suitable function space X and carefully defining $\mathrm{Dom}(A)$ to take the boundary conditions into account. (C_0) semigroup theory has proven useful in the context of differential equations, stochastic processes, approximation theory, mathematical physics, numerical functional analysis, nonlinear analysis, and so on.

While much of the theory of (C_0) semigroups and its applications was being worked out, the theory of (single) positive operators on Banach lattices (or on ordered Banach spaces) was undergoing intense development. Starting in the late 1970s these two subjects merged, and a beautiful theory of (C_0) semigroups of positive operators on Banach lattices (or positive semigroups, for short) emerged. The important contributors to this theory include the Tübingen group (i.e., the authors of the book under review), C. J. K. Batty, O. Bratteli, E. B. Davies, T. Kato, D. W. Robinson, and many others.

A number of recent books have been written about (C_0) semigroups (e.g. Davies [2], Fattorini [3], Goldstein [4], Pazy [8]), and about positive operators (e.g., Aliprantis-Burkinshaw [1], Schaefer [9], Zaanen [11]). Of these, Davies' book [2] is notable for its nice treatment of some of the modern topics in positive semigroup theory, but the book by Nagel and his eight coauthors is the first systematic treatment of these new theoretical developments and their applications.

Four main themes seem to emerge in this book: (I) characterization, (II) spectral theory, (III) asymptotics, and (IV) applications. We shall briefly indicate a sampling of the sort of results in each of these areas.

(I) CHARACTERIZATION. From formulas (2) and (3) it follows that T is a positive semigroup iff $(\lambda - A)^{-1}$ is a positive operator for all large positive λ. But one prefers a characterization of positive semigroups directly in terms of the generators rather than their resolvents. In 1973 T. Kato proved a distributional inequality involving the Laplacian (namely $\Delta|f| \geq (\mathrm{sgn}\,f)\Delta f$ for functions f in $L^2(\mathbf{R}^n)$) which quickly became an important tool for studying singular Schrödinger operators and other objects. This inequality is intimately related to the positivity of the heat semigroup $\{e^{t\Delta}\}$ on $L^2(\mathbf{R}^n)$; and inequalities of the form $A|f| \geq (\mathrm{sgn}\,f)Af$ for operators A on Banach lattices are nowadays called *Kato inequalities*. On some special Banach lattices, such as $C(K)$ (where K is a compact Hausdorff space), Kato's inequality for the generator characterizes the positivity of the semigroup. The general result, due to W. Arendt and A. Schep, is as follows.

THEOREM. *Let A generate a (C_0) semigroup T on a Banach lattice X. Then T is positive iff* (i) $\langle (\operatorname{sgn} f) A f, \phi \rangle \leqslant \langle |f|, A^* \phi \rangle$ *for all $f \in \operatorname{Dom}(A)$ and all $0 \leqslant \phi \in \operatorname{Dom}(A^*)$, and* (ii) *there is a set of positive linear functionals M in $\operatorname{Dom}(A^*)$ and a positive constant λ such that $A^* \phi \leqslant \lambda \phi$ for each $\phi \in M$ and whenever $0 \leqslant f \in X$, $f \neq 0$, there is a $\phi \in M$ such that $\langle f, \phi \rangle > 0$.*

Another beautiful, unexpected result is H. Lotz's theorem. If A generates a (C_0) semigroup on $L^\infty(\Omega, \Sigma, \mu)$ (or more generally on a Banach space with both the Grothendieck and Dunford-Pettis properties), then A is bounded.

(II) SPECTRAL THEORY. If A generates T, the *spectral bound* of A is given by $s(A) = \sup\{\operatorname{Re} \lambda : \lambda \in \sigma(A)\}$, while the *growth bound* of T is $\omega(A) = \inf\{w \in \mathbf{R} : \|T(t)\| \leqslant M_w e^{wt}$ for all $t \geqslant 0$ and some $M_w \geqslant 1\}$. Then $-\infty \leqslant s(A) \leqslant w(A) < \infty$, and each of the two $\leqslant$'s can be either equality or strict inequality. For various reasons it is important to know when $s(A) = \omega(A)$. This fails to hold for general positive semigroups on Banach lattices. But it does hold for positive semigroups on $X = L^p(\Omega, \Sigma, \mu)$ for $p \in \{1, 2, \infty\}$ (with the $p = \infty$ case including $C(K)$ for K compact and $C_0(\Omega)$ for Ω locally compact). The three proofs for the cases $p = 1, 2, \infty$ are all quite different, and the case of $X = L^p$ for $1 < p < \infty$, $p \neq 2$, is open.

We mention one of the many possible examples of a modern descendent of Perron-Frobenius theory. This is due to R. Derndinger and G. Greiner.

THEOREM. *Let A generate a positive semigroup. Let $\sigma_b(A) = \{\lambda \in \sigma(A) : \operatorname{Re} \lambda = s(A)\}$ be the portion of $\sigma(A)$ on the boundary line $\{\operatorname{Re} \lambda = s(A)\}$. If $\|(\lambda - s(A))(\lambda - A)^{-1}\|$ remains bounded as $\lambda \to s(A)^+$ or if $s(A)$ is a pole of $(\lambda - A)^{-1}$, then $\sigma_b(A)$ is cyclic in the sense that if $\alpha \in \mathbf{R}$ and $s(A) + i\alpha \in \sigma_b(A)$, then $s(A) + im\alpha \in \sigma_b(A)$ for all integers m.*

(III) ASYMPTOTICS. Let T be a positive semigroup on a Banach lattice. Then $s(A) < 0$ iff $\lim_{t \to \infty} e^{\delta t} \|T(A)f\| = 0$ for some $\delta > 0$ and all $f \in \operatorname{Dom}(A)$ iff A is invertible and $-A^{-1}$ is positive. Suppose that T is uniformly bounded, $\omega(A) = 0$, and $t \to T(t)$ is norm continuous for t sufficiently large. Then $\|T(t)f\| \to 0$ as $t \to \infty$ for all $f \in X$ iff the adjoint A^* is injective. When X is reflexive, this is equivalent to A being injective. In this last case, when A isn't injective, $\|T(t)f - Pf\| \to 0$ as $t \to \infty$ for all $f \in X$, where P is the positive projection onto the kernel of A along the range of A.

To illustrate how rapidly the field of positive semigroups is growing, rather than give an application from the book we shall present a newer one from mathematical biology.

(IV) APPLICATIONS. Very nice applications of positive semigroups to cell biology can be found in the works of the Amsterdam group (cf. e.g. [6, 7]) and Webb's book [10]. We shall give a result from a recent paper of Greiner and Nagel [5] which is based on work of Metz and Diekmann [7]. The growth of a cell population is described by the following model.

$$
(4) \quad
\begin{cases}
\dfrac{\partial}{\partial t} u(t, x) = -\dfrac{\partial}{\partial x}[g(x)u(t, x)] - \mu(x)u(t, x) \\
\qquad\qquad -b(x)u(t, x) + 4b(2x)u(t, 2x), \\
u(t, \alpha/2) = 0, \qquad u(0, x) = u_0(x).
\end{cases}
$$

Here $\alpha/2 > 0$ is the minimal cell size, β $(> \alpha)$ is the maximal cell size, $g \in C^1$ describes the growth rate (and $0 < \varepsilon_1 \leqslant g(x) \leqslant \varepsilon_2 < \infty$ for all x), μ and b $(\in C[\alpha/2, \beta])$ describe the death and division rates (with a cell being able to divide into two cells of equal size), $b(x) > 0$ for $\alpha < x < \beta$ and $b(x) = 0$ otherwise, and $2g(x) \geqslant g(2x)$ for all $x \in [\alpha/2, \beta/2]$. The last assmption (i.e., $2g(x) \geqslant g(2x)$) means that two offspring cells grow at least as fast as the parent cell.

Let $X = L^1[\alpha/2, \beta]$, $A_0 f = -gf' - mf$ where $m = g' + \mu + b$, and $f \in \text{Dom}(A_0)$ iff f is absolutely continuous on $[\alpha/2, \beta]$ and $f(\alpha/2) = 0$. Define B by $Bf(x) = 4b(2x)f(2x)$ or 0, according as $x \leqslant \beta/2$ or $x > \beta/2$. Then this problem (4) takes the form (1) with $A = A_0 + B$ (and B is a bounded perturbation of the generator A_0). A generates a positive semigroup T on X, and Metz and Diekmann [7] and Greiner and Nagel [5] study A and T in detail. The most interesting results concern the asymptotics.

Let

$$\xi(\lambda) = \int_{\alpha/2}^{1/2} \frac{4b(2s)}{g(s)} \exp\left[-\int_s^{2s} \frac{m(t) + \lambda}{g(t)}\, dt\right] ds.$$

Then the equation $\xi(\lambda) = 1$ has a unique real solution λ_0, and $\lambda_0 = s(A)$. All solutions of (4) tend to zero exponentially fast as $t \to \infty$ iff $\xi(0) < 1$. So consider the delicate case $\xi(0) = 1$, which makes $s(A) = \omega(A) = 0$. In this case 0 is an eigenvalue of A.

If $2g(x) > g(2x)$ for some x, then $\|T(t) - P\| \to 0$ as $t \to \infty$, where P is the positive projection onto the one-dimensional kernel of A. On the other hand, if $2g(x) = g(2x)$ for all x, then

$$\sigma(A) \cap i\mathbf{R} = \{im\delta\colon m = 0, \pm 1, \pm 2, \dots\},$$

where $\delta = 2\pi[\int_{\alpha/2}^{\alpha} g(t)^{-1}\, dt]^{-1} > 0$. Moreover, $T(t)$ converges strongly as $t \to \infty$ to a rotation semigroup of period $\int_{\alpha/2}^{\alpha} g(t)^{-1}\, dt$.

Since the field of positive semigroups and its applications is growing so rapidly, it would be impossible to produce a "final statement" of it at this time. But the book under review is a state-of-the-art presentation. It is the perfect vehicle to facilitate seminars on positive semigroups in the near future. Its contents are fresh and new and not available in any other single source at this time. Even though different chapters were written by different authors in various combinations, the book holds together nicely and has a stylistic as well as intellectual consistency.

Springer-Verlag describes its Lecture Notes in Mathematics series with the statement: "This series reports new developments in mathematical research and teaching—quickly, informally, and at a high level." This is a very accurate description of the book by Nagel et al.

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JEROME A. GOLDSTEIN

Tilings and patterns, by Branko Grünbaum and G. C. Shephard. W. H. Freeman and Company, New York, 1987, ix + 700 pp., $59.95. ISBN 0-7176-1193-1.

From time immemorial artisans and artists have constructed ingenious tilings and ornaments using repeated motives. This is demonstrated in the introduction of the beautiful volume under review by numerous examples from widely separated cultures. However, the importance of tilings and patterns in crystallography and some related branches of science was recognized only towards the end of the last century. From this time on many crystallographers, chemists, physicists, architects, engineers, and mathematicians have been working in this field. Although they accumulated a vast literature in books and periodicals, "much effort has been wasted duplicating previously known results." When the authors started collecting material for this book, they were surprised to find "how little about tilings and patterns is known," and how many errors were made because of "badly formulated definitions and lack of rigor."

For more than a decade the authors were busy critically revising the earlier results and making significant contributions to the theory of tilings and ornaments in a series of papers of their own. Their effort is crowned by the unique comprehensive monograph *Tilings and patterns*, which lays a solid foundation for one of the most attractive fields in geometry.

The book gives evidence of the sound didactic sense of the authors. The introduction of new concepts is carefully prepared, often supported by convincing intuitive arguments, and most formal definitions are richly illustrated by figures or some other means. The exposition is informal, but always clear and exact. Most sections contain carefully selected exercises, which often