will find several new and interesting considerations discussed. I certainly did. I noticed only a few typos. The bibliography is reasonable, but might be criticized for lacking more current references to the work of western scholars. However this would be a petty criticism to make in light of the author's personal circumstances.

From 1979 until just recently, Mark Freidlin struggled under severe restrictions on his academic activity imposed by the Soviet authorities. He is a refusenik, his applications to emigrate having been denied and having brought him into disfavor with the authorities. Both he and his wife were unemployed during this period, barred from participation in academic activities, receiving little of their mail and quite possibly denied access to university library facilities. It is a wonder that he was able to continue to produce timely and high-quality scientific work and of such size and scope as the book being reviewed here. This book represents not only an impressive contribution to the mathematical literature, but a monument to human courage and dignity. In March, 1987 he was finally granted permission to emigrate and will be taking a faculty position in this country.

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MARTIN DAY

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 17, Number 2, October 1987

0273-0979/87 $1.00 + $.25 per page

Applied nonlinear analysis, by Jean-Pierre Aubin and Ivar Ekeland. John Wiley and Sons, New York, Chichester, Brisbane, Toronto, and Singapore, xi + 518 pp., $52.50. ISBN 0-471-05998-6

From its very beginnings, two main themes have dominated analysis: the solution of equations and the study of (restricted) minima. They were never far apart, because necessary conditions for a minimum often appear in the form of

equations, and, conversely, many of the interesting equations represent first-order necessary conditions for certain minimization problems. Many of the other topics of analysis are also closely related to these first two. Implicit and inverse function theorems as well as fixed point and open mapping theorems and topological degree theory provide sufficient conditions for the existence of solutions of equations and of their perturbations. In addition, open mapping theorems are also used to eliminate candidates for a restricted minimum from competition and, by way of contradiction, to establish necessary conditions for a restricted minimum.

The theory of restricted minima—nowadays referred to as optimization—has in its "classical" form only involved equality restrictions. This was the case with the calculus of variations as well as with finite-dimensional problems. The few exceptions—such as Newton's treatment of the surface of least resistance that involved functions with nonnegative derivatives [**2**, p. 658]—by their very rarity and use of special devices only accentuated the general stress on equalities and equations. The first significant departure from this framework occurred during the second world war when the study of problems of economic and logistical nature—to a much greater extent than in physics—increasingly involved inequality restrictions. The simplest of such problems, those of linear programming, consist in minimizing $\varphi_0(z)$ subject to the inequalities $\varphi_i(z) \leqslant 0$ for $i = 1, \ldots, m$, where $\varphi = (\varphi_0, \ldots, \varphi_m)$: $R^k \to R^{m+1}$ is an affine function. Equivalently, by introducing "slack variables" $y_i \geqslant 0$, one seeks to minimize $f_0(x)$ subject to $f_i(x) = 0$ for $i = 1, \ldots, m$, where now

$$x = (z, y) = (z_1, \ldots, z_k, y_0, \ldots, y_m) \in C := R^k \times [0, \infty[^{m+1} \subset R^n,$$

$$f_i(x) = \varphi_i(z) + y_i \quad \text{for } i = 0, 1, \ldots, m,\ f = (f_0, \ldots, f_m).$$

Geometrically, this corresponds to defining the 0th component in R^{m+1} as vertical and searching for $\mathbf{x} \in R^n$ such that $f(\mathbf{x})$ is the lowest point on the 0th axis that belongs to $f(C)$. Since C is convex and f affine, the set $f(C)$ is convex, and it is clear that $f(\mathbf{x})$ lies on its boundary. Thus, by the convex separation theorem, there exists a "Lagrange multiplier" $\lambda = (\lambda_0, \ldots, \lambda_m) \in R^{m+1}$ such that $\lambda \neq 0$, $\lambda_i \geqslant 0$ and $\mathbf{x}$ minimizes the scalar product $\langle \lambda, f(x) \rangle$ over C. Exactly the same argument applies if, instead of assuming that f is affine and C is as indicated above, we postulate that $f(C)$ is convex; e.g., if we assume that the set $K \subset R^k$ and the functions φ_i: $K \to R$ for $i = 0, \ldots, m$ are convex. This last problem, no longer linear, is the more general convex programming problem, which has the property that any local solution is also global and first-order necessary conditions are also sufficient. Linear and convex programming problems, involving inequality constraints, are particular examples of optimization problems with inclusion constraints of the form $\varphi(x) \in A$ which appear ever more frequently in the theory and applications of optimization.

Convexity theory permeates many branches of analysis and in particular of optimization theory, even when the problems themselves are nonconvex. Thus the existence theorems in the calculus of variations and optimal control apply to "generalized trajectories" and "relaxed controls" obtained by replacing the set of "admissible" derivatives of the state functions at a point by its closed

convex hull. Some partial differential equations, especially of the elliptic type, can be studied as the Euler-Lagrange equations of a variational problem defined by a convex functional. Convexity theory also gives rise to one of the first examples of set-valued derivatives of nondifferentiable functions. If $C \subset R^n$ and f: $C \to R$ are convex then for every interior point $x \in C$ we can define the subdifferential $\partial f(x)$. This convex set consists of all vectors p such that $z \to f(z) - \langle p, z \rangle$ achieves its minimum at x, and $\partial f(x)$ becomes the singlet $\{f'(x)\}$ if f is Fréchet differentiable at x. The subdifferential of a convex function possesses many of the properties of the Fréchet derivative: thus, if x_0 in the interior of C minimizes f over C then $0 \in \partial f(x_0)$; and $f(x) - f(x_0) = \langle p, x - x_0 \rangle$ for some z on the segment $[x_0, x]$ and some $p \in \partial f(z)$. The inclusion $0 \in \partial f(x_0)$, whose solutions x_0 yield the minimum, is just one example of inclusion relations that are replacing equations in a number of applications that include partial differential equations and multidimensional variational problems [**1**].

The mapping $x \to \partial f(x)$ (for f a convex function) is an example of a set-valued function (also called a multifunction or a correspondence). If X and Y are given sets and $F(x)$ is a subset of Y for every $x \in X$ then F is referred to as a set-valued function or map from X to Y; it is called strict if $F(x) \neq \varnothing$ for all $x \in X$. F can be treated as an ordinary function from X to 2^Y but it is often more convenient to view it in the light of the original definition. Set-valued maps increasingly make their appearance in various branches of analysis. The first existence theorems in optimal control were derived from consideration of differential inclusions. As an example, consider the problem of minimizing $h_0(x(1))$ over the set of all absolutely continuous x: $[0,1] \to R^n$ such that

$$\dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. in } [0,1],$$

$$x(0) = 0 \quad \text{and} \quad h_1(x(1)) = 0 \in R^m,$$

where u: $[0,1] \to U \subset R^k$ can be chosen arbitrarily. If we set $F(t,x) := f(t,x,U)$ then the differential equation above is obviously equivalent to the differential inclusion $\dot{x}(t) \in F(t, x(t))$ a.e. While the original problem often has no minimizing solution even under very "nice" conditions, the relaxed problem, with $\dot{x}(t) \in F(t, x(t))$ replaced by $\dot{x}(t) \in \hat{F}(t, x(t))$, where $\hat{F}(t,x)$ is the convex hull of $F(t,x)$, has a minimizing solution under fairly general conditions, e.g., if there exists a feasible solution, f and h_i are continuous and bounded and U is compact. Thus the set-valued mapping $(t,x) \to \hat{F}(t,x)$ and the differential inclusion $\dot{x}(t) \in \hat{F}(t, x(t))$ a.e. become important tools of the optimal control of differential equations.

As was just mentioned, the differential equation $\dot{x}(t) = f(t, x(t), u(t))$ a.e. for some u: $[0,1] \to U$ is equivalent to the differential inclusion $\dot{x}(t) \in F(t, x(t))$ a.e. However, in many settings (e.g. in the study of necessary conditions), it is desirable to restrict $u(\cdot)$ to be measurable and thus to search for a measurable u: $[0,1] \to U$ satisfying the inclusion

$$u(t) \in \{v \in U \mid \dot{x}(t) = f(t, x(t), v)\} \quad \text{a.e in } [0,1].$$

This brings up the important problem of the existence of particular types of selections of set-valued mappings. In the most general case of an arbitrary strict set-valued mapping F between arbitrary sets X and Y, the resolution of the problem hinges on the axiom of choice. In less general settings, we have continuous and measurable selection theorems. For example, if X is metric and Y a Banach space then a theorem of Michael ensures that there exists a continuous selection $f\colon X \to Y$ of a set-valued mapping F from X to Y whenever each $F(x)$ is closed and convex and F is lower semicontinuous (i.e., for every $x_0 \in X$, $y_0 \in F(x_0)$ and any sequence (x_i) in X converging to x_0 there exists a sequence $y_i \in F(x_i)$ converging to y_0); if X and Y are metric and compact, X has a positive Radon measure μ defined on it, and F is μ-measurable (i.e., $\{x \in X \mid F(x) \cap A \neq \varnothing\}$ is μ-measurable for each closed $A \subset Y$), then there exists a μ-measurable selection f of F.

The theory of optimization, and in particular that of optimal control, also gave rise to nonsmooth analysis, which attempts to extend the methods and results of the differential calculus to nondifferentiable functions. Derivatives of nondifferentiable functions have long been used in the sense of the theory of distributions. However, the latter treats functions as elements of the class of linear functionals on a particular topological vector space. In this setting, the global properties of the functions come to the fore and the local properties are largely neglected. On the other hand, the stress of nonsmooth analysis is on the local behavior of the functions. We have already mentioned the subdifferential of convex analysis as one generalization of the concept of the derivative. A more general concept is Clarke's generalized gradient of a real-valued function on a Banach space, which acquires particular importance for locally Lipschitzian functions. If f is such a function in some neighborhood of a point $x \in R^n$, then one way to define its generalized gradient $\partial f(x)$ is to set

$$\partial^\varepsilon f(x) := \overline{\operatorname{co}}\{f'(z) \mid |z - x| \leqslant \varepsilon, f'(z) \text{ exists}\},$$

$$\partial f(x) := \bigcap_{\varepsilon > 0} \partial^\varepsilon f(x),$$

where $\overline{\operatorname{co}}$ is the symbol for the closed convex hull. (If φ is convex then the generalized gradient coincides with the subdifferential.) A similar definition yields Clarke's generalized Jacobian of a locally Lipschitzian $f\colon V \subset R^n \to R^m$. Still more general definitions, and their extensions to functions on or between Banach spaces, are various derivate containers **[6–8]** based on derivatives of uniform C^1 approximations to the given function in the neighborhood of x, or Ioffe's, Kruger's and Mordukhovič's approximate subdifferentials **[3–5]**. A typical result involving many set-valued derivatives is the inverse function theorem stating that if all elements of a generalized derivative of f at x are invertible with a uniform bound on the norms of the inverses, then f is a local homeomorphism with a Lipschitzian inverse.

A standard tool in deriving existence theorems or in the study of necessary conditions is a compactness argument. However, in many cases when that argument is inapplicable, it has been found that the existence of an "approximate" minimum suffices. The following "ε-variational principle" of Ekeland, barely 13 years old, has found many applications in several fields,

including the calculus of variations, optimal control and Hamiltonian systems. Let X be a complete metric space, U: $X \to]-\infty, \infty]$ a lower semicontinuous function bounded below and not identically ∞, $\varepsilon > 0$, and $U(x_\varepsilon) \leqslant \inf U + \varepsilon$. Then for every $k > 0$ there exists some $y_\varepsilon \in X$ such that

$$U(y_\varepsilon) \leqslant U(x_\varepsilon), \quad d(x_\varepsilon, y_\varepsilon) \leqslant 1/k, \quad U(y_\varepsilon) < U(x) + k\varepsilon d(x, y_\varepsilon) \text{ if } x \neq y_\varepsilon.$$

In the special case when X is a Banach space and U finite and Gâteaux differentiable at y_ε, it follows that $\|U'(y_\varepsilon)\|_* \leqslant k\varepsilon$, $\| \ \|_*$ denoting the norm in X^*.

The book of Aubin and Ekeland goes beyond most of the above topics in an attempt to present a picture of those parts of nonlinear analysis of greatest interest to the authors, of use in applications, and to which they had contributed important results. Their declared goal is to introduce students to nonlinear analysis "without going through the whole of linear analysis as a preliminary." They further state in the preface: "The ideas of nonlinear analysis are simple, their proofs direct, and their applications clear. No more prerequisites are needed than the elementary theory of Hilbert spaces; indeed, many results are most interesting in Euclidean spaces."

The statement about the prerequisites should not be interpreted literally. From the very beginning (Background Notes) the authors use the concepts and the results of the theory of measure and integration and presuppose acquaintance with basic concepts and properties of locally convex spaces (convex separation theorem, transpose of a linear operator, Banach contraction theorem, etc.). However, once these and similar qualifications are made, the book appears to be entirely self-contained, even when presenting very recent and advanced results. The basic framework involves functions between infinite-dimensional Banach spaces but, at times, either finite-dimensional or purely topological spaces make their appearance.

Chapter 2, on smooth analysis (dealing with C^r functions), has Newton's method for solving equations as a recurring theme. The method is used, in the conventional way, to prove the inverse function theorem, which then serves, using Milnor's approach, to derive various versions of Brouwer's fixed point theorem and Morese's lemma on singularities of a function. The study of bifurcating solutions of the equation $f(x, \lambda) = 0$ is then followed by a proof of Thom's transversality theorem. The latter is derived using Smale's approach based on his generalization of the Brown-Sard theorem to Banach manifolds. The chapter ends with Smale's "continuous" version—in the form of a differential equation—of Newton's method for solving the equation $f(x) = 0$ in R^n.

Chapter 3 is devoted to set-valued maps. If F is a set-valued map from X to Y then its image is $\bigcup_{x \in X} F(x)$ and its graph Graph(F) is $\{(x, y) \mid x \in X, y \in F(x)\}$; the map F is said to have property P if Graph(F) has that property. For X and Y Banach spaces, the authors study set-valued closed convex maps F and, in particular, convex processes, which are set-valued maps F whose graphs are convex cones so that

$$F(x_1) + F(x_2) \subset F(x_1 + x_2), \qquad F(\lambda x) = \lambda F(x) \text{ for } \lambda > 0.$$

Convex processes are set-valued analogues of linear operators. Among the theorems the authors derive are: any strict closed convex process F between Banach spaces is Lipschitzian, i.e., there exists some $k > 0$ such that, for all x_1 and x_2, $F(x_2)$ is in the $kd(x_1, x_2)$-neighborhood of $F(x_1)$; and generalizations to set-valued maps of theorems about positive matrices such as those of Perron-Frobenius and von Neumann-Kemeny. Chapter 6, on solutions of inclusions, includes a discussion of game theory, of Ky Fan's inequality, and of fixed point theorems for set-valued maps [existence of x satisfying $x \in F(x)$] such as Kakutani's, which are applied to a Walras model of the economy. The chapter concludes with a study of monotone maps in a Hilbert space [such that $\langle (y - q), (x - p) \rangle \geqslant 0$ whenever $q \in F(p)$, $y \in F(x)$] and of some related differential inclusions.

Convex analysis and convex optimization are discussed in Chapter 4. There, again, set-valued mappings are a basic tool to the extent that even an ordinary convex function $x \to \varphi(x)$ is often identified with the set-valued mapping $x \to \varphi_+(x) := \varphi(x) + [0, \infty[$. The authors focus on the study of the "marginal function" $y \to W(y) := \inf_{x \in X} V(x, y)$ corresponding to a convex $V(\cdot, \cdot)$ and on the dependence of the minimizers x_y of $x \to V(x, y)$ on y. In their study of differentiability properties the authors stress the geometric rather than the purely analytic concepts and approaches. Thus they first define the tangent and normal cones to convex sets and then define derivatives of a set-valued convex function F in terms of the tangent cones to the graph of F. Specifically, if $y_0 \in F(x_0)$ then the derivative of F at (x_0, y_0) is defined as the set-valued map whose graph is the tangent cone to Graph(F) at (x_0, y_0). The chapter introduces the conjugate function

$$p \to V^*(p) := \sup_{x \in X} [\langle p, x \rangle - V(x)]$$

of a convex lower semicontinuous V (which generalizes the Legendre transform) and uses conjugate functions in the study of Lagrange multipliers for restricted minimization problems and of the regularity of their solutions. The authors also study Lagrangians and Hamiltonians of convex problems that generalize those of the calculus of variations.

Chapter 7, on nonsmooth analysis, is dominated by the task of proving an inverse function theorem for a set-valued map F from X to Y without convexity assumptions. This theorem provides sufficient conditions for the existence of a solution x to the inclusion $y \in F(x)$ when $y_0 \in F(x_0)$ and y is in some neighborhood of y_0. A basic assumption is that the derivative $CF(x_0, y_0)$ has as its image all of Y. This derivative (a generalization of the one in the convex case) is defined as the closed convex process from X to Y whose graph is Clarke's tangent cone $C_K(x_0, y_0)$ to $K := \text{Graph}(F)$ at (x_0, y_0), where $C_K(z_0)$ is the set of all v such that

$$\lim_{\substack{z \to z_0,\ z \in K \\ h \to 0^+}} \frac{1}{h} d_K(z + hv) = 0$$

and $d_K(z)$ is the distance of z to K.

Chapter 5 deals with the ε-variational principle of Ekeland that we have stated before. The applications include an existence theorem for minimum of a lower semicontinuous Gâteaux differentiable function based on a weakened version of the Palais-Smale condition (C) and a strengthened version of a "mountain pass theorem" of Ambrosetti and Rabinowitz. Other applications include a theorem on the Fréchet differentiability on a residual set of a convex lower semicontinuous function on an open convex subset of a smooth Banach space, and a "generic" uniqueness and existence theorem for solutions v of the equation $F(x, v) = U(x)$ applying to all values of the parameter x in a residual set. [A residual set is a countable intersection of dense open sets.]

Finally, Chapter 8 applies the methods of Chapter 5 to the study of the existence of periodic solutions of Hamiltonian systems.

A fair number of the concepts and results presented in this book are due to the authors. The various concepts of derivatives of set-valued mappings (in Chapters 4 and 7) are due to Aubin, as is the set-valued inverse function theorem of Chapter 7 and an alternate model of the Walras equilibrium (Chapter 6). Ekeland's results in this book include: the ε-variational principle; jointly with Lebourg, the "generic" theorems on Fréchet differentiability and existence and uniqueness of solutions in Chapter 5; and many of the results on Hamiltonian systems in Chapter 8 (some of them obtained jointly with F. H. Clarke, who apparently initiated this line of investigation). The results first appearing in this book include theorems of Chapter 5 based on the authors' weak version of conditions (C) of Palais and Smale and, in particular, the strengthened version of the Ambrosetti-Rabinowitz theorem.

This book differs from most books on analysis by the pervasive use of set-valued mappings which, together with the ε-variational principle, are a principal tool of investigation. This probably reflects the influence of convex analysis whose methods and concepts are extended by the authors to non-smooth (nonconvex, non-C^1) analysis. This also gives the book a certain kind of unity despite the relative diversity of the topics that are brought together in it.

The authors, with prior books in their bibliographies, are good expositors. Each chapter starts with a glimpse into the basic results to follow and, frequently, with motivating comments. The proofs are often ingenious improvements on the original ones. The notes at the end of the book, grouped according to chapters and even sections, provide interesting information on the origins and evolution of many topics. On the other hand, the book would be much easier to follow if it contained a list of notations or, as an alternative, a much expanded index. As in any publication of this size, a few typographical errors remained undetected (but can usually be easily corrected by the reader). A random sample includes: p. 2, (2) [$x \in k$]; p. 58 [$x \to f(\lambda, x)$ on the second line below (1)]; p. 145 [X^* to X at the end of Corollary 26]; p. 240, l. 4 [$\mathcal{T}$ instead of C]; p. 297, l. 3* [(y, q)]. On p. 21, l. 6, a further subsequence must be chosen to ensure that $\lim x_{n_k}(\omega) = 0$ a.e.

This book is clearly an important contribution to the literature on the subject. In many ways it is a pioneering endeavor, and it offers a large number of tools and results for the advanced student as well as for the mature researcher.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 17, Number 2, October 1987

0273-0979/87 $1.00 + $.25 per page

Completely bounded maps and dilations, by Vern I. Paulsen. Pitman Research Notes in Mathematics, vol. 146, Longman Scientific and Technical, Essex, and John Wiley and Sons, New York, 1986, 187 pp., $38.95. ISBN 0-582-98896-9

This monograph uses C^*-algebraic techniques to study operator-theoretic problems. In particular, it uses the theory of completely positive and completely contractive maps to study dilations of operators. If T is a bounded linear operator on a Hilbert space H, then a dilation of T is a bounded linear operator S on a Hilbert space K containing H such that $Tx = PSx$ for each x in H, where P is the orthogonal projection of K onto H. Sometimes information about T can be obtained by dilating T to a "nice" operator S, using known facts about S, and then compressing back to H. Let $L(H)$ denote the algebra of all bounded linear operators on Hilbert space H. The two earliest dilation theorems are due to Naimark [3] and Sz.-Nagy [5]. Naimark proved that a regular, positive, $L(H)$-valued measure on a compact Hausdorff space can be dilated to be a spectral measure. Sz.-Nagy proved that if T belongs to $L(H)$ with the norm of T less than or equal to one, then T can be dilated to a unitary U such that $T^n x = PU^n x$ for all $n \geqslant 1$ and all x in H. Sz.-Nagy used this to prove von Neumann's inequality: If the norm of T is less than or equal to one and p is a polynomial, then $\|p(T)\| \leqslant \|p\|_\infty$, where $\|p\|_\infty$ denotes the uniform norm of p on the unit circle. Dilation theorems of various types are now standard in operator theory. See the book of Foiaş and Sz.-Nagy [6] or Halmos' Problem Book [2].

In 1955 W. F. Stinespring introduced a C^*-algebraic approach to dilation theory and used it to prove Naimark's theorem [4]. Besides the applications to