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Functional integration and partial differential equations, by Mark Freidlin, Annals of Mathematics Studies, vol. 109, Princeton University Press, Princeton, 1985, ix + 545 pp., $60.00 cloth, $19.95 paper. ISBN 0-691-8354-1

This book develops and explores certain classes of problems in partial differential equations by means of some fundamental connections with the theory of diffusion processes and stochastic differential equations. The typical PDE here is an elliptic or parabolic problem based on a second-order operator of the form

$$L = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i}, \tag{1}$$

with $a = [a_{ij}]$ symmetric, nonnegative definite.

The connection between linear PDEs and stochastic processes goes back many years, at least to the work of A. N. Kolmogorov in the 1930s. Both subjects have benefited from the interaction. Many people who deal with PDEs are aware of at least some of these classical connections. It is perhaps not so widely known however that certain asymptotic and singular perturbation questions, and even nonlinear problems, can be studied in this way as well.

For those not familiar with these classical connections, consider the familiar initial value problem for the heat equation in $\mathbf{R}^d$: find

$$u(\cdot,\cdot) \in C([0,\infty) \times \mathbf{R}^d) \cap C^{1,2}((0,\infty) \times \mathbf{R}^d)$$

which solves

$$u_t(t,x) = \tfrac{1}{2}\Delta u(t,x) \text{ for } t > 0; \quad u(0,x) = \varphi(x). \tag{2}$$

Given a bounded $\varphi \in C(\mathbf{R}^d)$, the unique solution is given by

$$u(t,x) = \int p(t,x,y)\varphi(y)\, dy \tag{3}$$

where $p(t, x, y)$ is the *heat kernel*:

$$p(t,x,y) = (2\pi t)^{-d/2} e^{-|y-x|^2/2t}.$$

The probabilist, on the other hand, knows the heat kernel as the "transition density for Brownian motion" in $\mathbf{R}^d$. A Brownian motion is a mapping $\omega \to \beta_\omega(\cdot)$ taking the elements ω of a probability space $(\Omega, \mathscr{F}, P)$ to continuous functions $\beta(\cdot) \in C([0,\infty), \mathbf{R}^d)$. It qualifies to be called a Brownian motion (starting at a given $x_0 \in \mathbf{R}^d$) when the measure induced on $\mathscr{C} = C([0,\infty), \mathbf{R}^d)$,

$$W_{x_0}(G) = P[\{\omega : \beta_\omega(\ ) \in G\}] \text{ for measurable } G \subseteq \mathscr{C},$$

is the particular measure called Wiener measure. It would be difficult to give a thorough description here; see [3]. However the connection with the heat kernel is exhibited in the special case $G = \{\beta(\cdot): \beta(t) \in \Gamma\}$, $\Gamma \subseteq \mathbf{R}^d$:

$$P[\{\omega : \beta_\omega(t) \in \Gamma\}] = \text{“}P_{x_0}[\beta(t) \in \Gamma]\text{”}$$
$$= \int_\Gamma p(t, x_0, y)\, dy.$$

(I am using quotation marks to indicate the standard notation.) In particular

$$\text{“}E_{x_0}[\varphi(\beta(t))]\text{”} = \int_\Omega \varphi(\beta_\omega(t)) P(dw)$$
$$= \int_{\mathbf{R}^d} p(t, x_0, y)\varphi(y)\, dy$$
$$= u(t, x_0).$$

Thus we have expressed the solution u of the PDE (2) as the "expected value" or integral of a functional of Brownian motion:

$$u(t, x_0) = E_{x_0}[\Phi(\beta(\cdot))] \quad \text{where } \Phi(f(\cdot)) = \varphi(f(t)),\ f \in \mathscr{C}. \tag{4}$$

Similar representations are possible in much wider generality. To replace the $\frac{1}{2}\Delta$ in (2) by the more general L of (1), one replaces the Brownian motion $\beta(t)$ by the solution $x(t)$ of the stochastic integral equation

$$x(t) = x_0 + \int_0^t b(x(s))\, ds + \int_0^t \sigma(x(s))\, d\beta(s). \tag{5}$$

The relation between the matrices $\sigma(\cdot)$ here and $a(\cdot)$ in (1) is $a(\cdot) = \sigma(\cdot)\sigma(\cdot)^{\mathrm{T}}$. With appropriate Lipschitz continuity and growth conditions on b and σ, (5) has a unique solution. (This is the subject of stochastic differential equations.) The solution $x(t)$ is another random continuous function, $\omega \to x_\omega(\cdot)$. Using the same functional Φ as in (4),

$$u(t, x_0) = E_{x_0}[\Phi(x(\cdot))] \tag{6}$$
$$= \int_\Omega \varphi(x_\omega(t)) P(d\omega)$$

gives the functional integral representation of the solution to

$$u_t = Lu \text{ for } t > 0; \qquad u(0, x) = \varphi(x),$$

provided the solution exists.

More general PDEs based on the same operator L have similar functional integral representations using the same process $x(t)$ but different functionals Φ. Another example, with the associated PDE for $u = E[\Phi(x(\cdot))]$, is

$$\begin{aligned} &\Phi(x(\cdot)) = \varphi(x(t)) e^{\int_0^t c(x(s))\, ds}, \\ &u_t = Lu + c \cdot u; \quad u(0, x) = \varphi(x). \end{aligned} \tag{7}$$

For elliptic problems in a domain $D \subseteq \mathbf{R}^d$, use the first exit time $\tau = \inf\{t > 0: x(t) \notin D\}$ to define the functional

$$\Phi(x(\cdot)) = \varphi(x(\tau)).$$

Then $u(x_0) = E[\Phi(x(\cdot))]$ will agree with the solution of the Dirichlet problem

$$Lu = 0 \text{ in } D; \quad u(x) = \varphi(x),\ x \in \partial D, \tag{8}$$

provided this is well posed in a classical sense. Several other linear problems also admit such representation of their solutions. Of course for these PDE problems to be well posed in a classical sense some other hypotheses are required, things like smoothness of the coefficients of L and of ∂D, and boundedness of D. An important observation is that the functional integral $u = E[\Phi(x(\cdot))]$ is well defined under only the weaker hypotheses needed for (5). For instance $a(\cdot)$ in (1) may be only nonnegative definite. Thus the functional integral representation provides a natural notion of "weak" solution and the theory of stochastic differential equations is a powerful set of tools for its study.

Freidlin's book explores the application of these ideas to numerous classical linear problems as well as nonlinear and asymptotic questions. In addition to formulation and proof of theoretical results, numerous examples are considered which illustrate the pathologies possible in nonclassical situations. The book as a whole provides a masterful and extensive demonstration of the techniques and ideas used in this approach to PDE problems. To be sure, a lot of what these methods yield about the PDE problems can be done just as well with more traditional PDE methods. On the other hand, the functional integral approach has a compelling intuitive appeal. Moreover in some problems, especially asymptotic ones, the functional integral approach seems to come closer to the heart of the problem and provide rigor beyond its analytical counterpart.

Chapter 1 is a nice survey of stochastic differential equations and related matters, including basic Wentzell-Freidlin large deviation theory. Though fairly extensive in its coverage, I doubt that the reader with no previous background will be able to appreciate the rest of the book based on the exposition in this chapter alone. Rather he will need first to study an entry level text on SDEs, such as Friedman [**3**]. For the reader who does have some background in SDEs, Chapter 1 may well be helpful by way of review, as well as alerting him to important aspects of the theory which may be missing in his background. He can then fill himself in from the other references, such as [**6**].

Chapter 2 covers linear problems for nondegenerate L, i.e. positive definite $a(\cdot)$. Much of this is either contained in or a natural extension of the treatments in other texts. For instance the exterior cone sufficient condition for regularity of points in ∂D (for the Dirichlet problem (8)) is discussed. The connection between uniqueness for the exterior Dirichlet problem and recurrence of $x(t)$ is developed, with stochastic Lyapunov function sufficient conditions, such as formulated by R. Z. Hasminskii [**4**]. Also discussed are problems with Neumann-type boundary conditions and their representation in terms of reflected diffusions and the local time process on ∂D. I am not aware of any other treatment of this topic in book form. Freidlin's exposition is fascinating in particular as he explains the probabilistic interpretation of classical solvability conditions on the Neumann boundary data. The inclusion of this is timely since there has been increased interest in these issues recently; see [**5**] and [**7**] for instance.

Chapter 3 discusses degenerate problems, and does so in more depth than I have seen in other texts. The decomposition of ∂D into the different components necessary for an accurate statement of boundary value problems is described rather carefully. General results on regularity of the functional integral solutions are developed. Several examples are presented which illustrate such phenomena as discontinuous solutions even for problems with C^∞ coefficients and boundary conditions.

Functional integral representations have proven quite fruitful in the study of various asymptotic questions. There has been a lot of recent work on problems in which L involves a small parameter, $\varepsilon \downarrow 0$,

$$L^\varepsilon = \varepsilon L_1 + L_0,$$

and one is interested in the ε-asymptotics of the solutions of various PDE problems associated with L^ε. Many such problems have been proposed as models in applied contexts. The functional integral representations allow these problems to be studied in terms of the ε-asymptotics of the diffusion process $x^\varepsilon(t)$ associated with L^ε. Exactly what asymptotic phenomena are involved depends on the particular problem. Sometimes convergence of $x^\varepsilon(t) \to x^0(t)$ (where $x^0(t)$ is the process associated with L_0) on compact time intervals is all that is involved. Other cases involve a second "fast" time scale whose asymptotic effect is to produce some ergodic "averaging" so that the limit $u^0 = \lim_{\varepsilon \downarrow 0} u^\varepsilon$ solves a reduced problem associated with an operator $\overline{L}_0$, which is L_0 with some spatial dependence appropriately averaged out.

A third class of asymptotic problems consists of ones in which the effect of εL_0 must "overcome" a contrary influence of L_0. The PDE approaches to these problems often involve such techniques as matched asymptotic expansions and WKB-style representations. On the probabilistic side the central ideas come from the subject of "large deviations." Freidlin, with A. D. Wentzell, pioneered much of this subject [**2**]. The probabilistic and WKB-style analyses frequently parallel each other and often both are illuminated as a result. For instance the "eiconal" equation which emerges from the WKB formalism typically is the Hamilton-Jacobi equation for a variational problem coming out of the large deviations analysis. The latter, and the role played by the variational problem, is valid even in those cases where singularities occur in the "solution" of the eiconal equation, preventing a classical solution. (The notion of "viscosity solution" seems to be bridging this gap from the PDE side; see [**1**].) Though this particular issue is not discussed in the book, I mention it to counter the misconception that functional integral techniques do nothing more than provide new proofs of old PDE results.

Chapter 4 of Freidlin's book considers numerous cases of the asymptotic problems that I have alluded to above. In addition to discussing some of the important general types of problems, he also analyzes a number of particular examples which illustrate what can happen in other situations, such as degenerate perturbing noise (i.e. second-order part of L_1, having less than full rank).

The final three chapters deal with nonlinear problems. Chapter 5 is a discussion of how functional integral techniques can be applied to quasilinear problems, in which the coefficients a, b of (1) as well as c of (7) are allowed to depend on the solution u. In essence the idea is to think of the solution as a fixed point of the mapping which takes u to the coefficients of L, thence to the functional integral representation of u. Such a fixed point is a reasonable notion of a weak solution. It turns out that the space of Lipschitz continuous functions is a natural class of functions in which to formulate this. One can then develop sufficient conditions on the coefficient functions for there to exist a unique such weak solution (globally) as well as sufficient conditions for it to be a classical solution. Most of this chapter seems to be taken from Freidlin's own work, published in the late 60s.

The final two chapters of the book cover an interesting application of the asymptotic methods of Chapter 4 to the study of wave-like properties of solutions to

$$u_t = \tfrac{1}{2}u_{xx} + f(u), \tag{9}$$

where $f \geqslant 0$ only if $0 \leqslant u \leqslant 1$. One might think of $u(t, x)$ as the concentration of a diffusing substance with the nonlinear term $f(u)$ describing the rate of generation of the substance as the result of a density-dependent reaction. It turns out that such equations can have "traveling wave" solutions, $u(t, x) = V(x - \alpha t)$ where the function $V(\cdot)$ describes the wave shape and α its velocity. Of course one can study such solutions by plugging $u = V(x - \alpha t)$ into (9), obtaining a nonlinear eigenvalue problem for the pair $(\alpha, V(\cdot))$. However an asymptotic rescaling allows the velocity α to be isolated. To wit, $u^\varepsilon(t, x) = u(t/\varepsilon, x/\varepsilon)$ satisfies

$$\frac{\partial}{\partial t}u^\varepsilon = \frac{\varepsilon}{2}u^\varepsilon_{xx} + \frac{1}{\varepsilon}f(u^\varepsilon). \tag{10}$$

Now if the "wave" is an advancing region of saturation, i.e. $u \sim 1$ for $x \ll \alpha t$, $u \sim 0$ for $x \gg \alpha t$, or simply

$$\lim_{z \to -\infty} V(z) = 1, \qquad \lim_{z \to +\infty} V(z) = 0,$$

then since $u^\varepsilon(t, x) = V(\frac{x - \alpha t}{\varepsilon})$,

$$u^\varepsilon(t, x) \to \begin{cases} 1 & \text{if } x < \alpha t \\ 0 & \text{if } x > \alpha t \end{cases} \qquad \text{as } \varepsilon \downarrow 0. \tag{11}$$

Applying techniques of large deviations analysis to (10) one can study the speed α by means of property (11). This technique allows one to isolate α from the wave form $V(\cdot)$. Chapter 6 studies (10). Some very interesting phenomena can occur if spatial inhomogeneity is allowed in (10), such as the appearance of new saturated regions at a positive distance from previously saturated points. Such features end up being described as properties of the variational problems arising in the large deviations asymptotic analysis. Chapter 7 considers the influence of averaging effects in random or periodic media producing an effective or asymptotic velocity in similar problems. Again the general discussions of these chapters is supplemented by numerous specific examples which illustrate various possible special phenomena.

There is a wealth of material in this book. I have mentioned the large number of illustrative examples already. Theoretical ideas and methods are well demonstrated in the formal proofs as well. I think that even most experts will find several new and interesting considerations discussed. I certainly did. I noticed only a few typos. The bibliography is reasonable, but might be criticized for lacking more current references to the work of western scholars. However this would be a petty criticism to make in light of the author's personal circumstances.

From 1979 until just recently, Mark Freidlin struggled under severe restrictions on his academic activity imposed by the Soviet authorities. He is a refusenik, his applications to emigrate having been denied and having brought him into disfavor with the authorities. Both he and his wife were unemployed during this period, barred from participation in academic activities, receiving little of their mail and quite possibly denied access to university library facilities. It is a wonder that he was able to continue to produce timely and high-quality scientific work and of such size and scope as the book being reviewed here. This book represents not only an impressive contribution to the mathematical literature, but a monument to human courage and dignity. In March, 1987 he was finally granted permission to emigrate and will be taking a faculty position in this country.

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Applied nonlinear analysis, by Jean-Pierre Aubin and Ivar Ekeland. John Wiley and Sons, New York, Chichester, Brisbane, Toronto, and Singapore, xi + 518 pp., $52.50. ISBN 0-471-05998-6

From its very beginnings, two main themes have dominated analysis: the solution of equations and the study of (restricted) minima. They were never far apart, because necessary conditions for a minimum often appear in the form of